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MATHEMATICS MAGAZINE

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VOLUME 45 • NOVEMBER 1972 • NUMBER 5

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MATHEMATICS MAGAZINE

GERHARD N. WOLLAN, *EDITOR*

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THE LONG JUMP MIRACLE OF MEXICO CITY

M. N. BREARLEY, RAAF Academy, Victoria, Australia

1. Introduction. Did a miracle occur at the 1968 Olympic Games in Mexico City? The Games record for the long jump (Harlan, [**1**]), which had crept up since 1904 by five successive increments totalling 0.76 *m*, was raised by 0.80 *m* in a single mighty leap. In his first (and only) attempt, R. Beamon of the U.S.A. lifted the record from 8.10 *m* (26 ft $6\frac{7}{8}$ in.) to 8.90 *m* (29 ft $2\frac{1}{2}$ in.), an increase of 9.9 %. The enormity of this feat may be gauged by noting that a similar improvement in the present mile world record time of about 3 min 51 sec. would cut it by nearly 23 seconds to about 3 min 28 sec.

It has been suggested that Beamon's leap owed much to the reduced air drag resulting from the lower air density at the high altitude (7400 ft) of Mexico City. We seek a mathematical model of the long jump which will enable the influence of the Mexico City altitude to be assessed quantitatively.

2. Setting up the mathematical model. The distance achieved in a long jump is affected by many factors such as spring off the board, the action of the free leg at take-off, rotation of the body by a hitch-kick, and landing technique. To model all such aspects would be difficult; fortunately it is unnecessary for the present purpose.

After take-off the athlete's center of mass G describes a path which can be investigated like that of any projectile moving through a resisting medium. For our purpose it will plainly suffice to compare the horizontal range of a projectile *in vacuo* with its range when similarly projected in a medium of known density. The effects of air densities at various altitudes can thus be compared without considering details of the long jump technique.

In traditonal notation the equation of motion of the athlete after take-off is

$$m\frac{d\boldsymbol{V}}{dt} = m\boldsymbol{g} + \boldsymbol{D}, \tag{1}$$

where $\boldsymbol{V}$ is the vector velocity of G at any instant and $\boldsymbol{D}$ is the air drag on the athlete. The direction of $\boldsymbol{D}$ is opposite to that of $\boldsymbol{V}$, and its magnitude D is known from experimental work of Nonweiler [**2**] to be given by

$$D = k\rho V^2,$$

where ρ is the air density, $V = |\boldsymbol{V}|$, and k is constant for a fixed body posture. Hence

$$\boldsymbol{D} = -k\rho V^2 \hat{\boldsymbol{V}},$$

and on division by m equation (1) becomes

(2) $$\frac{dV}{dt} = g - KV^2 \hat{V},$$

where

(3) $$K = k\rho/m.$$

It is convenient to refer the flight path of G to the traditional axes Ox, Oy, where O is the position of G at take-off. At time t after take-off, let

u, v = velocity components of G parallel to Ox, Oy.

The initial conditions of the flight are then

(4) $$t = 0,\ x = y = 0,\ u = u_0,\ v = v_0,$$

where u_0, v_0 may for the present remain arbitrary.

The nonlinear differential equation (2) has no exact solution in closed form. Though it can be solved approximately to an accuracy sufficient for the present purpose, for the sake of brevity a less orthodox approach will be used.

3. The energy dissipated by air resistance. The theory of projectiles in the absence of air resistance yields the familiar results

(5a, b, c) $$u = u_0, \qquad v = v_0 - gt, \qquad y = v_0 t - \tfrac{1}{2}gt^2,$$

(6a, b, c) $$v = \pm(v_0^2 - 2gy)^{\frac{1}{2}} \qquad h = \tfrac{1}{2}v_0^2/g, \qquad R_0 = 2u_0v_0/g,$$

where h and R_0 are the height and range attained relative to a horizontal plane. From the principle of conservation of energy it readily follows that

(7) $$V = (V_0^2 - 2gy)^{\frac{1}{2}},$$

where

(8) $$V_0^2 = u_0^2 + v_0^2.$$

In the presence of air resistance the energy per unit mass dissipated during the total flight time t_1 is

(9) $$E = \int_0^{t_1} KV^2\,\hat{V}\cdot V\,dt = \int KV^3v^{-1}dy,$$

where the latter integral is evaluated over the total domain of y. It may be verified that the change in velocity caused by air resistance is very small (averaging about 1.3% in a typical case), so that E may be found very accurately by using (6a) and (7) to approximate the integrand in (9). It is found that

$$E = \int_0^h KV^3(v_0^2 - 2gy)^{-\frac{1}{2}}dy - \int_h^0 KV^3(v_0^2 - 2gy)^{-\frac{1}{2}}dy,$$

$$= 2K\int_0^h (V_0^2 - 2gy)^{3/2}(v_0^2 - 2gy)^{-\frac{1}{2}}dy.$$

On changing to the new variable

$$z = (v_0^2 - 2gy)/u_0^2$$

it is readily found with the aid of (6b) and (8) that

$$E = (Ku_0^4/g)\int_0^{r^2} z^{-\frac{1}{2}}(1+z)^{3/2}\,dz, \tag{10}$$

where $r = v_0/u_0$. This integral can be converted to an incomplete Beta function by putting $z = w/(1-w)$. It may also be evaluated accurately by expanding the integrand in powers of z, leading to the approximation

$$E = (Ku_0^4/g)(2r + r^3). \tag{11}$$

The error in this curtailed form of the series (whose terms alternate in sign after the second) is less than the magnitude $0.15r^5$ of the third term. In a typical case this error is found to be about 0.3% of the value of E.

With the aid of (6c), equation (11) yields

$$E = K(u_0^2 + \tfrac{1}{2}v_0^2)R_0. \tag{12}$$

The initial kinetic energy per unit mass, namely

$$T_0 = \tfrac{1}{2}V_0^2 = \tfrac{1}{2}(u_0^2 + v_0^2) \tag{13}$$

is finally reduced by air resistance to an approximate value

$$T_1 = T_0 - E. \tag{14}$$

4. The effect of air resistance on the length of jump. Let the *range* R of the jump be defined in terms of the coordinates (x, y) of G as

$$R = \text{the value of } x \text{ for which } y = 0, \qquad t > 0.$$

The range is slightly less than the measured length of a jump, chiefly because (i) the athlete's feet are forward of G when they strike the ground, (ii) contact with the ground does not occur until $y < 0$, which is at a later time than that for which R is calculated. Our object is to compare jump lengths for different air densities; if this is done by comparing R values instead, the error incurred is clearly very small.

The range R_0 *in vacuo* of a projectile with initial energy T_0 as in equation (13) is given by

$$R_0 = 2u_0v_0/g. \tag{6c}$$

Its range R in air will be calculated as if it were launched *in vacuo* at the same angle of elevation but with initial energy T_1 as in equation (14). In place of (6c) we will then have

$$R = 2u_1v_1/g, \tag{15}$$

where

(16) $$u_1/u_0 = v_1/v_0 = (T_1/T_0)^{\frac{1}{2}}.$$

From (6c), (14), (15) and (16) it is readily seen that

$$R = [1 - (E/T_0)]R_0.$$

With the aid of (8), (12) and (13) we can write this as

(17) $$R = [1 - K\{1 + (u_0/V_0)^2\}R_0]R_0.$$

The increment in range which accompanies a decrease $-\delta K$ in K is seen from (17) to be

(18) $$\delta R = [1 + (u_0/V_0)^2]R_0^2(-\delta K).$$

Starting from the obvious fact that equation (10) overestimates the value of E defined in (9), it is easily seen that (18) yields an overestimate of δR. It is therefore suitable for gauging the maximum possible effect of a change in air resistance on the distance achieved in a long jump.

5. An example: Beamon's Mexico City jump. Nonweiler [**2**] lists values of $2k$ (which he calls *drag area*) for three cyclists at speeds comparable with that of a long-jumping athlete. For his Subject C (whose stature resembles that of Beamon) he lists for the touring and racing positions of the cyclist the respective values

(19a, b) $$k = 0.182\ m^2, \qquad k = 0.163\ m^2.$$

Considering the marked difference between the touring and racing postures which are pictured in Nonweiler's paper, the values of k in (19a, b) differ by surprisingly little. Over most of the long jump flight path an athlete's posture alters by no more than that of a cyclist between touring and racing positions; it is therefore a reasonable approximation to take k as constant throughout the jump. The value in equation (19a) for the touring position with cycle is taken as the appropriate one. Any error incurred will affect similarly the ranges calculated for air of different densities, and so the error in the difference of such ranges will be very small.

Let us consider a jump which *in vacuo* would have a range

(20) $$R_0 = 8.0\ m = 26 \text{ ft } 3.0 \text{ in.}$$

This would correspond to a measured jump length of about 9.0 m, or 29 ft 6.3 in.

For use in (13) we will take the estimated value

(21) $$u_0 = 9.45\ m\ s^{-1},$$

which corresponds to a sprinting speed of 100 yards in 9.6 seconds. (It may be verified that the final result does not depend critically on the value alloted to u_0, the main conclusion being unaltered if we change u_0 to a value equivalent to 100 yards in 11.0 seconds.) The corresponding values of v_0 and V_0 are found from (6c), (20), (21) and (8) to be

(22a, b) $$v_0 = 4.15\ m\ s^{-1}, \qquad V_0 = 10.32\ m\ s^{-1}.$$

The densities of air at sea-level and at the Mexico City altitude of 7400 ft (= 2256 *m*) are given by Gray [**3**, p. 3–61] as

(23a, b) $$\rho_1 = 1.225\ kg\ m^{-3}, \qquad \rho_2 = 0.984\ kg\ m^{-3}.$$

Taking the mass m of the athlete to be 80 *kg*, it follows from (3), (19a) and (23a, b) that the values of K at sea-level and 7400 ft are respectively

(24a, b) $$K_1 = 2.787 \times 10^{-3} m^{-1}, \qquad K_2 = 2.239 \times 10^{-3} m^{-1}.$$

The change in K accompanying a move from sea-level to Mexico City is therefore

(25) $$\delta K = K_2 - K_1 = -0.548 \times 10^{-3} m^{-1}.$$

The corresponding increment in range for the Beamon jump represented by equation (20) is found from (18), (22a, b) and (25) to be

(26) $$\delta R = 0.0645\ m = 2.5\ \text{in}.$$

This is an upper bound for the extra distance which Beamon gained from the lower air resistance in Mexico City.

6. Discussion. The distance increment of 2.5 inches in equation (26) is much smaller than the gain which athletics experts usually ascribe to reduced air drag. A more accurate (but longer) solution of equation (2) by successive approximations leads to a refined value of 0.7 inches for the gain δR in distance; it also shows that even the total removal of air resistance would add only about 4.6 inches to a sea-level jump of world record length. It is easily verified that the altitude of Mexico City causes a decrease of about 0.07% in the sea-level value of g, and that the resulting increment in a jump of 8.9 meters would be only about 0.25 inches.

Reduced air resistance enables a slightly greater forward speed to be achieved during the run-up and take-off. Slightly longer jumps at high altitude sites would result, but the effect is negligibly small for two reasons:

(i) The increase in sprinting speed caused by lowered air resistance is trivial. This statement is supported by the failure of sprinters to lower the world record for the 100 meters (I.A.A.F., [**4**]) at the Mexico City Olympic Games.

(ii) *Most* long jumpers do not leave the take-off board at the greatest sprinting speed of which they are capable. They prefer to sacrifice speed (and length of jump) rather than risk the "no-jump" penalty for over-stepping the board or the loss of spring which accompanies a take-off made from behind the board.

The foregoing work shows that it is pure myth that Beamon's great jump owed much to reduced air drag. This conclusion is supported by the failure of all other long jumpers to improve upon their "personal bests" at the Mexico City Games.

With the reduced air drag theory discredited, it is natural to seek other explanations of Beamon's remarkable jump. The writer believes that a clue is given by the italicized word "*Most*" in the above paragraph labelled (ii). Beamon was not one of the conservative majority of dedicated board strikers when he jumped in Mexico City. He is a sprinter capable of running 100 yards in 9.5 seconds, and he tried to reach his top speed during his approach. He gave scant attention to the take-off board and relied solely on his carefully measured run-up to ensure that he hit the board accurately; his whole effort went into the jump itself. By a statistical accident his take-off foot landed in perfect position on the board; the rest of the performance was the inevitable consequence of the world's finest long-jumper using a take-off speed which had never before been achieved. That Beamon himself knew it was a statistical miracle is evidenced by the fact that he did not even use his second and third jumps. If his first jump had failed he would presumably have reverted to a more conventional technique to lessen the risk of missing the board in his subsequent attempts.

The foregoing assessment suggests that the standard take-off board (dimensions $1.22\ m \times 0.2\ m \times 0.1\ m$) severely detracts from long jumping performances because its small width makes it hard to strike cleanly at top speed. A much wider board (dimensions $1.22\ m \times 0.5\ m \times 0.1\ m$, say) would encourage faster approaches and significantly improve long jumping performances. Retention of the present practice of measuring the length of the jump from the front of the board would provide a suitable bonus for the athlete with an accurate approach run.

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1. H. V. Harlan, History of the Olympic Games, Foster, London, 1964.
2. T. Nonweiler, The air resistance of racing cyclists, Coll. of Aero. Report no. 106 (1956).
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WALKING IN THE RAIN

MICHAEL A. B. DEAKIN, Monash University, Clayton, Australia

Abstract. We consider a simplified model of a man walking (or running) in the rain. We ask what speed he should adopt if he is to remain as dry as possible after traversing a fixed distance. Usually, it is in his best interest to run as fast as possible, but this is not always the case.

1. The problem. Parallel steady rain is falling. It is necessary for a man to walk (or run) through this from point A to point B. He has no umbrella and thus seeks to proceed at a speed (u) which minimizes the total amount of rain falling on him. We seek to find the optimum value of u.

We set up a model as follows. Let $\boldsymbol{i}$ be a unit vector in the direction $\overrightarrow{AB}$, let $\boldsymbol{k}$ be tions of Beamon's remarkable jump. The writer believes that a clue is given by the italicized word "*Most*" in the above paragraph labelled (ii). Beamon was not one of the conservative majority of dedicated board strikers when he jumped in Mexico City. He is a sprinter capable of running 100 yards in 9.5 seconds, and he tried to reach his top speed during his approach. He gave scant attention to the take-off board and relied solely on his carefully measured run-up to ensure that he hit the board accurately; his whole effort went into the jump itself. By a statistical accident his take-off foot landed in perfect position on the board; the rest of the performance was the inevitable consequence of the world's finest long-jumper using a take-off speed which had never before been achieved. That Beamon himself knew it was a statistical miracle is evidenced by the fact that he did not even use his second and third jumps. If his first jump had failed he would presumably have reverted to a more conventional technique to lessen the risk of missing the board in his subsequent attempts.

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1. The problem. Parallel steady rain is falling. It is necessary for a man to walk (or run) through this from point A to point B. He has no umbrella and thus seeks to proceed at a speed (u) which minimizes the total amount of rain falling on him. We seek to find the optimum value of u.

We set up a model as follows. Let $\boldsymbol{i}$ be a unit vector in the direction $\overrightarrow{AB}$, let $\boldsymbol{k}$ be a unit vector pointing upwards and let $\boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{i}$. The raindrops, supposed to be of uniform size, are falling at their terminal speed V_T and are being swept along by a horizontal wind whose velocity is $V_T(w\boldsymbol{i} + W\boldsymbol{j})$. Thus the velocity of the rain is given by:

$$V = V_T(w\boldsymbol{i} + W\boldsymbol{j} - \boldsymbol{k}). \tag{1}$$

The man's speed u can be expressed in terms of V_T by:

$$u = xV_T, \tag{2}$$

so that the relative velocity of the raindrops as seen by the man is:

$$V_{rel} = V_T\{(w - x)\boldsymbol{i} + W\boldsymbol{j} - \boldsymbol{k}\}. \tag{3}$$

The man himself we model as a cuboid getting wet on three of his six sides: the front (or back), one side (taken to be the right), and the top. These are taken to have areas A, ηA, εA respectively.

2. The wetness function. The amount of rain impinging on the top surface per unit time will be proportional to the area εA multiplied by the cosine of the angle between the normal to this area ($\boldsymbol{k}$) and the relative direction of the rain. This cosine has the value:

$$\frac{1}{\sqrt{(w-x)^2 + W^2 + 1}}.$$

Thus the amount of rain falling on the top surface is proportional to:

$$\frac{\varepsilon A}{\sqrt{(w-x)^2 + W^2 + 1}}.$$

Similarly, the amount landing on the right hand surface is proportional to:

$$\frac{\eta A W}{\sqrt{(w-x)^2 + W^2 + 1}}.$$

A slight complication occurs with rain impinging on the back or front. This provides a contribution:

$$\frac{A|w-x|}{\sqrt{(w-x)^2 + W^2 + 1}},$$

where the absolute value takes account of the fact that the man gets equally wet whether the rain comes toward him from the front or the back.

Thus the total amount of rain falling on the man per unit time is proportional to:

$$\frac{\phi + |w-x|}{\sqrt{n^2 + (w-x)^2}},$$

where

$$\phi = \varepsilon + \eta W \tag{4}$$

$$n^2 = 1 + W^2. \tag{5}$$

The total amount of rain falling on the man is proportional to the amount landing on him per unit time and inversely proportional to his speed — i.e., is proportional to:

$$F(x) = \frac{\phi + |w - x|}{x\sqrt{n^2 + (w - x)^2}}. \tag{6}$$

$F(x)$ will be termed the "wetness function". We seek to minimize $F(x)$.

3. Walking into the rain. If the rain is coming from the front of a stationary man, w will be negative. We find it more convenient in this case to redefine w to be positive. This slightly changes the expression for F which now reads:

$$F(x) = \frac{\phi + (w + x)}{x\sqrt{n^2 + (w + x)^2}}. \tag{7}$$

It may readily be proved that for $x > 0$ (the region of interest) $F(x)$ as given by (7) is monotonic decreasing. Thus $F(x)$ is minimized when x is made as large as possible. Thus one's best strategy if running into the rain is to run as fast as possible.

4. Walking away from the rain. The case $w > 0$ leads to a function $F(x)$ given directly by equation (6). $F'(x)$ has a discontinuity at $x = w$. At this speed, neither the man's back nor his front is getting wet. Above this speed, the man overtakes raindrops and so his front gets wet. Below this speed, he still has the rain at his back.

We readily discover that:

$$F(w) = \frac{\phi}{nw}, \tag{8}$$

$$F'(w-) = -\frac{1}{nw^2}(w + \phi), \tag{9}$$

$$F'(w+) = \frac{1}{nw^2}(w - \phi). \tag{10}$$

We note that $F'(w-)$ is always negative. $x = w$ will thus provide a relative minimum if $F(w+) > 0$, i.e., if $w > \phi$.

The cases $w < \phi$, $w > \phi$ will be considered separately below. We refer to them as Case 1 and Case 2 respectively.

5. Behavior of F for small x. When $x < w$, $F(x)$ is given by:

$$F(x) = \frac{\phi + w - x}{x\sqrt{n^2 + (w - x)^2}}.$$

We wish to ascertain whether this function has any turning points in the interval $(0, w)$. By differentiating F and setting $F'(x) = 0$, we arrive at the equation:

$$x^3 - (3w + 2\phi)x^2 + 3w(\phi + w)x - (n^2 + w^2)(\phi + w) = 0. \tag{11}$$

By Descartes' Rule of Signs, this cubic has either one or three positive roots and no negative roots. Setting $y = w - x$, we obtain:

$$y^3 + 2\phi y^2 - w\phi y + n^2(\phi + w) = 0. \tag{12}$$

By Descartes' Rule of Signs, this has one negative root and either two positive roots or no positive roots. The two positive roots occur, if at all, in the region of interest. They occur if and only if equation (12) has three roots. Applying the discriminant condition for cubics, we find that these roots occur if:

$$\left(\frac{\phi}{3}\right)^3\left(\frac{4\phi}{3} + w\right)^3 > \left[\frac{16}{27}\phi^3 + \frac{2}{3}w\phi^2 + n^2(\phi + w)\right]^2,$$

which reduces to:

$$\begin{aligned}\frac{1}{3}w^3\phi^3 > \frac{192}{729}\phi^6 + \frac{16}{243}w\phi^5 + \frac{32}{27}n^2\phi^4 + \frac{68}{27}n^2w\phi + \frac{4}{3}w^2n^2\phi^2 \\ + n^4\phi^2 + 2wn^4\phi + n^4w^2.\end{aligned} \tag{13}$$

A necessary condition for (13) to hold is:

$$\frac{1}{3}w^3\phi^3 > \frac{4}{3}w^2n^2\phi^2, \tag{14}$$

where the term on the right is the fifth of the right hand terms in inequality (13). Condition (14) reduces to:

$$w > \frac{4n^2}{\phi}$$

$$\text{i.e.,} \quad w > \frac{4(1 + W^2)}{\varepsilon + \eta W}.$$

Thus

$$w > 4\min\left(\frac{1 + W^2}{\varepsilon + \eta W}\right),$$

i.e.,

$$w > \frac{8}{\eta}\left(\sqrt{1 + \left(\frac{\varepsilon}{\eta}\right)^2} - \frac{\varepsilon}{\eta}\right). \tag{15}$$

Here and later we adopt the values:

$$\varepsilon \sim 0.06,\ \eta \sim 0.33,\ V_T \sim 20 \text{ m.p.h.} \tag{16}$$

The value of V_T was supplied by a meteorologist colleague. The other values were obtained by measurement of the author. (15) now reduces to

$$w > 20.1 \cdots,$$

i.e., for maxima and minima of the wetness function to exist in the region $(0, w)$, we require a wind whose component of velocity in the $\mathbf{i}$-direction is in excess of 400 mph. We ignore this possibility as unrealistic.

Thus the wetness function may be assumed to be monotonic decreasing on $(0, w)$.

6. Case 1. This case is typified by the condition $w < \phi$. That is to say a person standing in the rain facing in the $\mathbf{i}$-direction gets wetter on the top and right hand side than he does on the back.

For $x > w$, $F(x)$ is given by:

$$F(x) = \frac{\phi + x - w}{x\sqrt{n^2 + (x-w)^2}}. \tag{17}$$

An analysis similar to that of the previous section leads to the equation:

$$y^3 + 2\phi y^2 + w\phi y - n^2(w - \phi) = 0 \tag{18}$$

where $y = x - w$.

The region of interest is given by $y > 0$ and equation (18) has no roots in this region under the conditions applying to Case 1. Thus there are no maxima or minima of $F(x)$ when $x > w$ in this case.

$F(x)$ therefore is monotonic decreasing for all x and once again one's best strategy is to run as fast as possible.

7. Case 2. Case 2 is that for which a man standing in the rain facing in the $\mathbf{i}$-direction gets wetter on the back than he does on the top and right hand side. Equation (18) applies also to this case, but this time the final term is negative and a root exists. This is in accord with heuristic considerations, since $F(x) \to 0$ as $x \to \infty$. For this case $F(x)$ has the form shown in Figure 1.

There will be a unique point $x = X$ for which $F(X) = F(w)$. Setting $X = w + Y$ we find:

$$\phi^2 Y^3 + 2w\phi^2 Y^2 + (n^2\phi^2 + w^2\phi^2 - w^2 n^2)Y - 2w\phi n^2(w - \phi) = 0, \tag{19}$$

which, by Descartes' Rule of Signs, has only one positive root.

Speeds x lying in the range $w < x < X$ are to be avoided as they result in greater wetting than necessary. The best strategy to follow depends on the maximum speed of which the walker (or rather runner) is capable. If this is greater than X, he should

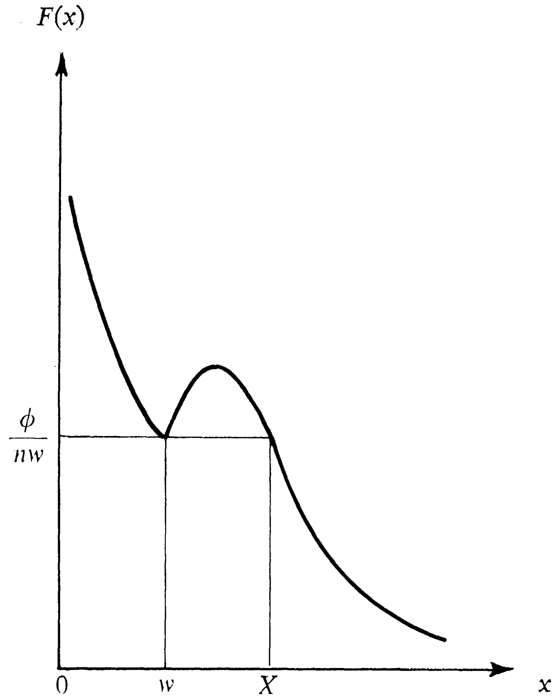

FIG. 1. Qualitative behavior of the function $F(x)$ in Case 2 ($w > \phi$).

run at his maximum speed. If, however, his maximum speed is less than X, he does better to reduce speed until he presents only his top and right as targets — i.e., to try to "walk between the drops".

Using the values given in equations (16) and assuming a maximum possible speed of 20 m.p.h. (i.e., $x = 1$), the situation was investigated by computer. For a range of values of W between 0,3, equation (19) was solved for a number of values of w in the range $(\phi, 1)$. That value of w for which the solution of (19) is equal to $1 - w$ was readily obtained. This value we term w_{crit}. It was thus possible to obtain a plot of w_{crit} against W. This is shown in Figure 2.

If $w > w_{crit}$, it is best to drop one's speed to correspond with that of the rain, unless, of course, $w > 1$, in which case once more it becomes advantageous to run as fast as possible.

In the case $\phi < w < w_{crit}$, the setting of x equal to w achieves a relative minimum which might be preferred to the actual minimum ($x = 1$) on the grounds of lesser effort.

8. Sensitivity analysis. Consider two alternative strategies. The naive strategy consists of setting x equal to its maximum value in every case; a more sophisticated strategy is outlined in this article. We compare the effects of these two strategies. The two strategies are identical except in the case

$$w_{crit} < w < 1,$$

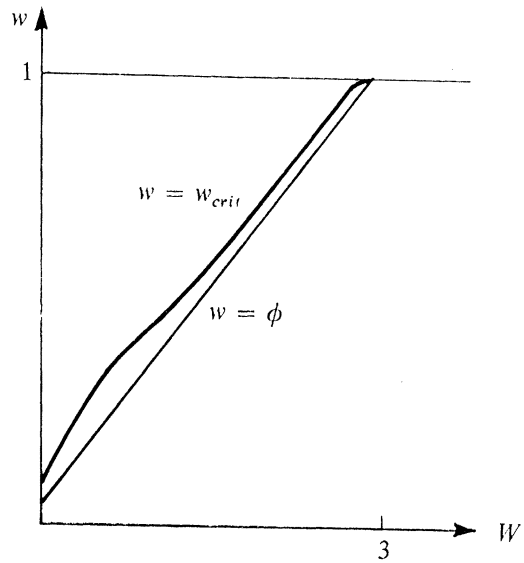

FIG. 2. Dependence of w_{crit} on W. The line $w = \phi$ is included for comparison.

and thus this case alone will be considered.

If the naive strategy is followed, the relevant wetness function is $F(1)$, but if the sophisticated strategy is pursued the wetness function is ϕ/nw, by equation (8). Let

$$R = \frac{nwF(1)}{\phi}. \tag{20}$$

R measures the penalty incurred if the naive strategy is followed. E.g., if $R = 2$, a man following the naive strategy gets twice as wet as his sophisticated counterpart, etc.

A computer analysis was carried out on the case investigated earlier. The results are shown in Figure 3. The contour line $R = 1$ is necessarily identical with the curve $w = w_{crit}$ of Figure 2. This provides a useful check on the accuracy of the calculation. The curves produced by the two methods checked to the limits of accuracy involved. (For better display of more interesting contour lines, only part of this curve has been shown in Figure 3.)

It will be seen that for small values of W, the penalty associated with the naive strategy is quite severe. R is substantially in excess of unity over a significant area of the $w - W$ plane.

As W increases, it becomes less important to follow the sophisticated policy, and for the larger values of W, R is very close to unity. E.g., when $W = 2$, R has a maximum of just over 1.025. These values correspond to gales of some considerable force, however, so that perhaps they are less relevant to the analysis.

The worst value of x to choose is that which gives rise to the relative maximum in the wetness function (see Figure 1). This is the cause of the maximal values of R. One is scarcely likely to choose a policy of walking or running so as to achieve

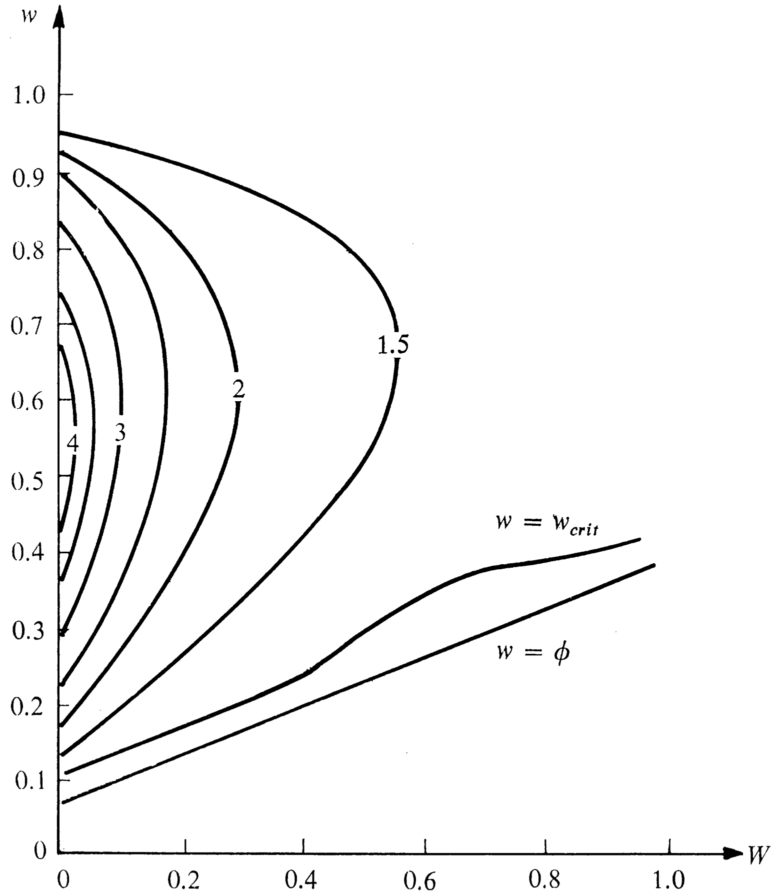

FIG. 3. Dependence of R on w, W. The contour $R = 1$ is made up of the curve $w = w_{crit}$ and the segment of the line $w = 1$ between $W = 0$ and $W = 2.85$.

this maximum, but one may achieve it inadvertently by running as fast as possible. It is this situation that produces maximal value of R for any given W.

9. Summary. We have analyzed a simple model, probably the simplest available, of a man walking in the rain. For this model, it is usually advantageous to run at maximum velocity. Such a strategy usually minimizes the extent of wetting. However, if the rain is such that a man standing in it and facing in the direction of travel is wet predominantly on his back, then there exists a range of wind-speeds for which this policy would not work. In such cases it is best to match one's speed to the relevant component of the rain's velocity. If the cross component of the wind is small, the naive policy of running as fast as possible can produce over four times the wetting resulting from the more sophisticated policy. As this component of the wind increases, the difference between the two strategies becomes less pronounced.

Acknowledgments: I wish to thank Mr. R. Petrou who drew the diagrams. I also acknowledge a valuable suggestion by the referee.

A SOLUTION OF LAPLACE'S EQUATION FOR A SEMI-INFINITE STRIP

CLARENCE R. EDSTROM, Air Force Institute of Technology

Broman [1, page 163] states the problem of finding a harmonic function subject to certain boundary conditions on the edges of a semi-infinite strip and he indicates that the Fourier sine transform is to be used to obtain the solution. However, some problems of this type can be solved by another method.

We consider the boundary value problem

$$\nabla^2 u = 0 \text{ for } x > 0, \qquad 0 < y < L \tag{1}$$

subject to the conditions

$$u(x,0) = u(0,y) = 0, \; u(x,L) = f(x), \text{ and } |u(x,y)| < M \tag{2}$$

where M is a constant. As this problem is stated, the separation of variables technique does not apply. However, if the function $f(x)$ is sufficiently well behaved, there exists a change of dependent variable such that the method of separation of variables can be applied in order to determine the new dependent variable.

We assume that there exists an integer N such that each of the derivatives of order $N + 1$ and $N + 2$ of the function $f(x)$ can be expressed as a linear combination of the function $f(x)$ and its first N derivatives. Since N may not be unique, we will select the smallest possible value for N. Next we assume the change of dependent variable from $u(x, y)$ to $U(x, y)$ given by

$$u(x,y) = U(x,y) + \sum_{n=0}^{N} G_n(y) f^{(n)}(x), \quad \left(f^{(n)}(x) = \frac{d^n f(x)}{dx^n} \right). \tag{3}$$

Substituting $u(x, y)$ from (3) into (1) we obtain

$$\nabla^2 U + \sum_{n=0}^{N} [G_n(y) f^{(n+2)}(x) + G_n''(y) f^{(n)}(x)] = 0$$

where the primes denote differentiation with respect to the argument of the function. Since we want $U(x, y)$ to satisfy

$$\nabla^2 U = 0 \text{ for } x > 0, \qquad 0 < y < L, \tag{4}$$

we will determine the functions $G_n(y)$ so that

$$\sum_{n=0}^{N} [G_n f^{(n+2)}(x) + G_n'' f^{(n)}(x)] = 0. \tag{5}$$

From the assumption of the existence of the integer N we can write

$$f^{(N+1)}(x) = \sum_{n=0}^{N} a_n f^{(n)}(x) \text{ and} \tag{6}$$

$$f^{(N+2)}(x) = \sum_{n=0}^{N} b_n f^{(n)}(x) \tag{7}$$

where the a_n and b_n are constants. Rearranging the terms of (5) and substituting for $f^{(N+1)}(x)$ and $f^{(N+2)}(x)$ from (6) and (7) we obtain

$$\begin{cases} [G_0'' + a_0 G_{N-1} + b_0 G_N]f(x) + [G_1'' + a_1 G_{N-1} + b_1 G_N]f'(x) \\ \quad + \sum_{n=2}^{N} [G_n'' + G_{n-2} + a_n G_{N-1} + b_n G_N] f^{(n)}(x) = 0. \end{cases} \tag{8}$$

Equation (8) will be satisfied if the functions $G_n(y)$ satisfy the system of linear homogeneous differential equations

$$\begin{cases} G_0'' + a_0 G_{N-1} + b_0 G_N = 0 \\ G_1'' + a_1 G_{N-1} + b_1 G_N = 0 \\ G_n'' + G_{n-2} + a_n G_{N-1} + b_n G_N = 0, \qquad n = 2, 3, 4, \cdots, N. \end{cases} \tag{9}$$

Also we want

$$U(x, 0) = U(x, L) = 0. \tag{10}$$

Hence we go back to (3) and choose

$$\begin{cases} G_n(0) = 0, \ n = 0, 1, 2, \cdots, N \text{ and} \\ G_0(L) = 1, \ G_n(L) = 0, \qquad n = 1, 2, 3, \cdots, N \end{cases} \tag{11}$$

as the boundary conditions for the system of equations (9).

Since we are dealing with two point boundary conditions, a solution of the system of equations (9) subject to the conditions (11) may not exist. If a solution does not exist, the change of dependent variable will have to be modified and we will consider this modification later in this paper.

The one remaining condition which is needed for the determination of $U(x, y)$ is obtained from (3) when $x = 0$. Thus we have

$$U(0, y) = -\sum_{n=0}^{N} G_n(y) f^{(n)}(0). \tag{12}$$

Therefore the boundary value problem in $U(x, y)$ consists of equation (4) subject to the conditions (10) and (12) with $|U(x, y)| < M_1$ where M_1 is a constant.

In order to illustrate this method we consider the boundary value problem consisting of (1) and (2) with

$$f(x) = x \exp(-x) \tag{13}$$

and $L \neq p\pi$ where p is an integer. Differentiating (13) we have

$$f'(x) = (1 - x)\exp(-x). \tag{14}$$

Also each of the derivatives $f''(x)$ and $f'''(x)$ can be expressed as a linear combination of $f(x)$ and $f'(x)$ so that $N = 1$. These linear combinations are

$$f''(x) = -f(x) - 2f'(x) \text{ and} \tag{15}$$

$$f'''(x) = 2f(x) + 3f'(x). \tag{16}$$

Comparing (15) and (16) with (6) and (7) where $N = 1$ we obtain

$$a_0 = -1,\ a_1 = -2,\ b_0 = 2, \text{ and } b_1 = 3. \tag{17}$$

Since we now know the a_n, b_n, and N, the system of equations (9) becomes

$$\begin{cases} G_0'' - G_0 + 2G_1 = 0 \\ G_1'' - 2G_0 + 3G_1 = 0 \end{cases} \tag{18}$$

and from (11) we obtain the boundary conditions

$$G_0(0) = G_1(0) = G_1(L) = 0 \text{ and } G_0(L) = 1. \tag{19}$$

The solution of the system of equations (18) subject to the conditions (19) is

$$\begin{cases} G_0(y) = \dfrac{\sin y}{\sin L} \text{ and} \\ G_1(y) = \dfrac{L \operatorname{ctn} L \sin y - y \cos y}{\sin L}. \end{cases} \tag{20}$$

From (20) we observe the necessity of the restriction which was imposed on L. If $L = p\pi$, we have no solution of the system of equations (18) subject to the conditions (19) and thus no change of dependent variable as given by (3).

Now substituting for $f(x)$, $f'(x)$, $G_0(y)$, and $G_1(y)$ from (13), (14), and (20) into (3) we obtain the change of dependent variable from $u(x, y)$ to $U(x, y)$ as

$$u(x, y) = U(x, y) + \frac{\exp(-x)}{\sin L}[(x + L \operatorname{ctn} L) \sin y - y \cos y]. \tag{21}$$

Therefore the boundary value problem for $U(x, y)$ becomes equation (4) subject to the conditions (10),

$$U(0, y) = \frac{y \cos y - L \operatorname{ctn} L \sin y}{\sin L}, \tag{22}$$

and $|U(x, y)| < M_1$. Then using the standard technique of separation of variables as given in Churchill [**2**, page 34], the solution is

$$U(x, y) = \frac{4\pi}{L^2} \sum_{n=1}^{\infty} \frac{n(-1)^{n+1} \exp(-n\pi x/L) \sin(n\pi y/L)}{[(n\pi/L)^2 - 1]^2}. \tag{23}$$

Hence substituting $U(x, y)$ from (23) into (21) yields a solution of the boundary value problem given by (1) and (2) with (13) when $L \neq p\pi$.

Now we will return to the original boundary value problem and consider the case in which there is no solution of the system of equations (9) subject to the conditions (11). In this case we will assume the change of dependent variable from $u(x,y)$ to $U(x,y)$ given by

$$u(x,y) = U(x,y) + H(y)F(x) + \sum_{n=0}^{N} H_n(y)f^{(n)}(x). \tag{24}$$

Substituting $u(x,y)$ from (24) into (1) we obtain

$$\nabla^2 U + H(y)F''(x) + H''(y)F(x) + \sum_{n=0}^{N} [H_n(y)f^{(n+2)}(x) + H_n''(y)f^{(n)}(x)] = 0.$$

As before we want $U(x,y)$ to satisfy (4) so we will determine the functions $H(y)$ and $H_n(y)$ so that

$$HF''(x) + H''F(x) + \sum_{n=0}^{N} [H_n f^{(n+2)}(x) + H_n'' f^{(n)}(x)] = 0. \tag{25}$$

Also we will choose the function $F(x)$ so as to satisfy

$$F''(x) = AF(x) + \sum_{n=0}^{N} c_n f^{(n)}(x) \tag{26}$$

where A and c_n are constants. Rearranging the terms of (25) and substituting for $f^{(N+1)}(x)$, $f^{(N+2)}(x)$, and $F''(x)$ from (6), (7), and (26) we obtain

$$\begin{cases} [H'' + AH]F(x) + [H_0'' + c_0 H + a_0 H_{N-1} + b_0 H_N]f(x) \\ \quad + [H_1'' + c_1 H + a_1 H_{N-1} + b_1 H_N]f'(x) \\ \quad + \sum_{n=2}^{N} [H_n'' + H_{n-2} + c_n H + a_n H_{N-1} + b_n H_N]f^{(n)}(x) = 0. \end{cases} \tag{27}$$

Equation (27) will be satisfied if $H(y)$ and $H_n(y)$ satisfy the system of linear homogeneous differential equations

$$\begin{cases} H'' + AH = 0 \\ H_0'' + c_0 H + a_0 H_{N-1} + b_0 H_N = 0 \\ H_1'' + c_1 H + a_1 H_{N-1} + b_1 H_N = 0 \\ H_n'' + H_{n-2} + c_n H + a_n H_{N-1} + b_n H_N = 0, \quad n = 2,3,4,\cdots,N. \end{cases} \tag{28}$$

Since we want the conditions (10), we go back to (24) and choose

$$\begin{cases} H(0) = H_n(0) = 0, \quad n = 0,1,2,\cdots,N \text{ and} \\ H_0(L) = 1 \text{ and } H(L) = H_n(L) = 0, \quad n = 1,2,3,\cdots,N \end{cases} \tag{29}$$

as the boundary conditions for the system of equations (28).

The remaining condition needed for the determination of $U(x, y)$ is obtained from (24) when $x = 0$. Thus we have

$$U(0, y) = -F(0)H(y) - \sum_{n=0}^{N} H_n(y)f^{(n)}(0). \tag{30}$$

Therefore the boundary value problem in $U(x, y)$ now consists of equation (4) subject to the conditions (10) and (30) with $|U(x, y)| < M_2$ where M_2 is a constant.

Now we return to the example and consider the case $L = p\pi$. Since any function $F(x)$ which satisfies (26) is admissible, we will choose

$$F(x) = x^2 \exp(-x). \tag{31}$$

Differentiating (31) twice and rewriting in the form of (26) we obtain

$$F''(x) = F(x) - 2f(x) + 2f'(x) \tag{32}$$

so that

$$A = 1, \ c_0 = -2, \text{ and } c_1 = 2. \tag{33}$$

Then substituting $N = 1$ and the values (17) and (33) into the system of equations (28) we have

$$\begin{cases} H'' + H = 0 \\ H_0'' - 2H - H_0 + 2H_1 = 0 \\ H_1'' + 2H - 2H_0 + 3H_1 = 0 \end{cases} \tag{34}$$

and from (29) we obtain the boundary conditions

$$H(0) = H_0(0) = H_1(0) = H(p\pi) = H_1(p\pi) = 0 \text{ and } H_0(p\pi) = 1. \tag{35}$$

A solution of the system of equations (34) subject to the conditions (35) is

$$\begin{cases} H(y) = \dfrac{(-1)^{p+1}\sin y}{2p\pi} \\ H_0(y) = \dfrac{(-1)^p}{2p\pi}[y^2 \sin y + 2y\cos y] \\ H_1(y) = \dfrac{(-1)^p y^2 \sin y}{2p\pi}. \end{cases} \tag{36}$$

Next substitute for $F(x)$, $f(x)$, $f'(x)$, $H(y)$, $H_0(y)$, and $H_1(y)$ from (31), (13), (14), and (36) into (24) in order to obtain the change of dependent variable from $u(x, y)$ to $U(x, y)$ as

$$u(x, y) = U(x, y) + \frac{(-1)^p \exp(-x)}{2p\pi}[2xy\cos y - (x^2 - y^2)\sin y]. \tag{37}$$

Hence the boundary value problem for $U(x, y)$ becomes equation (4) subject to the conditions (10) with $L = p\pi$,

$$U(0, y) = \frac{(-1)^{p+1} y^2 \sin y}{2p\pi}, \tag{38}$$

and $|U(x, y)| < M_2$. Again using the separation of variables technique, the solution is

$$U(x, y) = \frac{(-1)^{p+1}}{p\pi} \sum_{n=1}^{\infty} \exp\left(\frac{-nx}{p}\right) \sin\left(\frac{ny}{p}\right) \int_0^{p\pi} t^2 \sin t \sin\left(\frac{nt}{p}\right) dt. \tag{39}$$

Thus substituting (39) into (37) yields a solution of the boundary value problem given by (1) and (2) with (13) when $L = p\pi$.

If $p = 1$, then (39) becomes

$$U(x, y) = \left(\frac{\pi}{6} - \frac{1}{4\pi}\right) \exp(-x) \sin y + \frac{4}{\pi} \sum_{n=2}^{\infty} \frac{(-1)^{n+1} n \exp(-nx) \sin ny}{(n^2 - 1)^2}. \tag{40}$$

If p is any other integer, a corresponding form for $U(x, y)$ can be computed. Hence, the problem can be solved for any width strip.

References

1. A. Broman, Introduction to Partial Differential Equations from Fourier Series to Boundary-Value Problems, Addison-Wesley, Reading, Massachusetts, 1970.

2. R. V. Churchill, Fourier Series and Boundary Value Problems, 2nd ed., McGraw-Hill, New York, 1963.

THE gcd OF CERTAIN BINOMIAL COEFFICIENTS

JOE ALBREE, University of Tennessee, Knoxville

The following formula which involves some admittedly cumbersome calculations is a generalization of a problem from THE AMERICAN MATHEMATICAL MONTHLY [**1**]. Additionally, the proof presented here, even though it is a little longer, is less demanding of the background of the reader than that which appeared in the solution to the MONTHLY'S problem [**2**].

THEOREM. *For any positive integer* N,

$$p^r = \gcd\left\{\binom{N}{k} \text{ where } 1 \leq k \leq N \text{ and } (k, p) = 1\right\}, \tag{1}$$

where p *is prime,* r *is a positive integer and* $p^r \| N$.

Proof. Throughout, we shall let $N = p^r \cdot p_1^{r_1} \cdots p_m^{r_m}$, where p, p_i are distinct

Hence the boundary value problem for $U(x, y)$ becomes equation (4) subject to the conditions (10) with $L = p\pi$,

$$U(0, y) = \frac{(-1)^{p+1} y^2 \sin y}{2p\pi}, \tag{38}$$

and $|U(x, y)| < M_2$. Again using the separation of variables technique, the solution is

$$U(x, y) = \frac{(-1)^{p+1}}{p\pi} \sum_{n=1}^{\infty} \exp\left(\frac{-nx}{p}\right) \sin\left(\frac{ny}{p}\right) \int_0^{p\pi} t^2 \sin t \sin\left(\frac{nt}{p}\right) dt. \tag{39}$$

Thus substituting (39) into (37) yields a solution of the boundary value problem given by (1) and (2) with (13) when $L = p\pi$.

If $p = 1$, then (39) becomes

$$U(x, y) = \left(\frac{\pi}{6} - \frac{1}{4\pi}\right) \exp(-x) \sin y + \frac{4}{\pi} \sum_{n=2}^{\infty} \frac{(-1)^{n+1} n \exp(-nx) \sin ny}{(n^2 - 1)^2}. \tag{40}$$

If p is any other integer, a corresponding form for $U(x, y)$ can be computed. Hence, the problem can be solved for any width strip.

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THEOREM. *For any positive integer* N,

$$p^r = \gcd\left\{\binom{N}{k} \text{ where } 1 \leqq k \leqq N \text{ and } (k, p) = 1\right\}, \tag{1}$$

where p is prime, r is a positive integer and $p^r \| N$.

Proof. Throughout, we shall let $N = p^r \cdot p_1^{r_1} \cdots p_m^{r_m}$, where p, p_i are distinct primes and $r > 0$, $r_i > 0$, $i = 1, \cdots, m$. We shall see that $\binom{N}{p_i^{r_i}}$ are the important terms in (1). The formula

$$\binom{N}{k} = \frac{N \cdot (N-1) \cdots (N-k+1)}{k \cdot (k-1) \cdots 3 \cdot 2 \cdot 1} \tag{2}$$

will be used, and our proof will be accomplished in two steps: (a) p^r divides every $\binom{N}{k}$, $1 \leqq k \leqq N$ and $(k, p) = 1$; (b) each $\binom{N}{p_i^{r_i}}$ is relatively prime to that p_i. The idea in each part is simply to count certain factors in the numerator and denominator of (2) and then see that the appropriate factors either remain or are cancelled. (Note that $\binom{N}{1}$ is one of our binomial coefficients, so the gcd does divide N.)

Because $(k, p) = 1$, the first integer smaller than k which p divides is $k - j$ for some $j \geqq 1$; the xth such number is $k - j - (x-1)p$. If there are t of these multiples of p in the denominator of (2), the last one is p itself and hence

$$k - j - (t-1)\ p = p, \text{ which implies that } t = \frac{k-j}{p}.$$

The first integer less than N which p divides is $N - p$; the xth such integer is $N - xp$. Therefore, the tth such number is

$$N - tp = N - k + j \geqq N - k + 1,$$

and so p divides the same number of factors in the numerator and the denominator of (2). Now, we cancel these p's (not p^2, or p^3, etc.) and the factors of the cancelled numbers which remain we call the *reduced p coefficient*, $\begin{bmatrix} \frac{N-p}{p} \\ t \end{bmatrix}$, noting that $(N - tp)/p = (N-p)/p - t + 1$. We write

$$\begin{bmatrix} N \\ k \end{bmatrix} = X \cdot \begin{bmatrix} \frac{N-p}{p} \\ t \end{bmatrix}, \text{ where}$$

$$X = \frac{N \cdot (N-1) \cdots \left(\frac{N-p}{p}+1\right) \cdot \left(\frac{N-p}{p}-1\right) \cdots \left(\frac{N-tp}{p}+1\right) \cdot \left(\frac{N-tp}{p}-1\right) \cdots (N-k+1)}{k \cdot (k-1) \cdots (tp+1) \cdot (tp-1) \cdots (p+1) \cdot (p-1) \cdots 2 \cdot 1}.$$

Since p^r divides the numerator of X but is relatively prime to its denominator (we have left N unaltered in X's numerator and we have placed all of the multiples of p from the denominator of (2) in the denominator of the reduced p coefficient), p^r divides $\binom{N}{k}$ to complete (a).

We start (b) by pairing off the numerator and denominator of $\binom{N}{p_i^{r_i}}$:

$$\begin{matrix} N-1 & & N-h & & N-(p_i^{r_i}-1) \\ \updownarrow & \cdots & \updownarrow & \cdots & \updownarrow \\ 1 & & h & & p_i^{r_i}-1. \end{matrix} \tag{3}$$

The first integer less than N which is divisible by p_i^s, $1 \leqq s \leqq r_i - 1$, is $N - p_i^s$; the xth such number is $N - xp_i^s$ (the first integer smaller than N which is divisible by

$p_i^{s'}$, $s' \geqq r_i$, if such a number exists, is less than or equal to $N - p_i^{r_i}$, and consequently, because the smallest factor in the numerator is $N - p_i^{r_i} + 1$, is not in the numerator of $\binom{N}{p_i^{r_i}}$. We have $p_i^s \mid N - h$ iff $p_i^s \mid h$, and thus every power of p_i in the list (3) gets cancelled. But we have retained, undisturbed, N in the numerator and $p_i^{r_i}$ in the denominator of $\binom{N}{p_i^{r_i}}$; so we divide $p_i^{r_i}$ into N and (b) follows to complete the proof.

I am happy to record here my thanks to Professor R. M. McConnel for checking this work.

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THE NO-TOUCH PUZZLE AND SOME GENERALIZATIONS

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The object of the original no-touch puzzle was to place the first eight natural numbers in eight cells, related in various ways, so that adjacent numbers are never in related cells. The relationship of the cells in the original is indicated in Diagram 1.

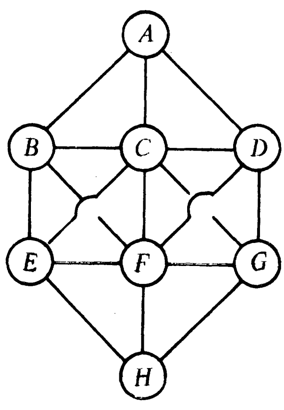

DIAGRAM 1

Clearly, one wants to place the numbers with the fewest adjacent numbers, namely 1 and 8, in the cells with the most neighbors, namely cells C and F. Once the 8 is placed in cell C and the 1 in cell F, we are forced to place the 2 in the three-cell A and the 7 in the three-cell H. The 3 must go in either cell E or cell G since any other vacant cell would be a neighbor of A which is occupied by 2. Similarly the 6 must go in cell B or D except that it cannot go in a cell which is a neighbor of the cell containing

$p_i^{s'}$, $s' \geqq r_i$, if such a number exists, is less than or equal to $N - p_i^{r_i}$, and consequently, because the smallest factor in the numerator is $N - p_i^{r_i} + 1$, is not in the numerator of $\binom{N}{p_i^{r_i}}$. We have $p_i^s \mid N - h$ iff $p_i^s \mid h$, and thus every power of p_i in the list (3) gets cancelled. But we have retained, undisturbed, N in the numerator and $p_i^{r_i}$ in the denominator of $\binom{N}{p_i^{r_i}}$; so we divide $p_i^{r_i}$ into N and (b) follows to complete the proof.

I am happy to record here my thanks to Professor R. M. McConnel for checking this work.

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THE NO-TOUCH PUZZLE AND SOME GENERALIZATIONS

D. K. COHOON, University of Minnesota

The object of the original no-touch puzzle was to place the first eight natural numbers in eight cells, related in various ways, so that adjacent numbers are never in related cells. The relationship of the cells in the original is indicated in Diagram 1.

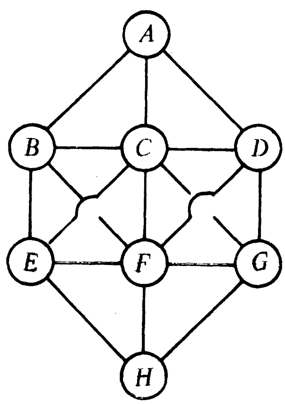

DIAGRAM 1

Clearly, one wants to place the numbers with the fewest adjacent numbers, namely 1 and 8, in the cells with the most neighbors, namely cells C and F. Once the 8 is placed in cell C and the 1 in cell F, we are forced to place the 2 in the three-cell A and the 7 in the three-cell H. The 3 must go in either cell E or cell G since any other vacant cell would be a neighbor of A which is occupied by 2. Similarly the 6 must go in cell B or D except that it cannot go in a cell which is a neighbor of the cell containing

3 since that would force 4 and 5 into neighboring cells. Thus, once the 3 is placed, the places of the other numbers are determined. Thus, there are only four solutions of the puzzle with 1 and 8 in the two six-cells.

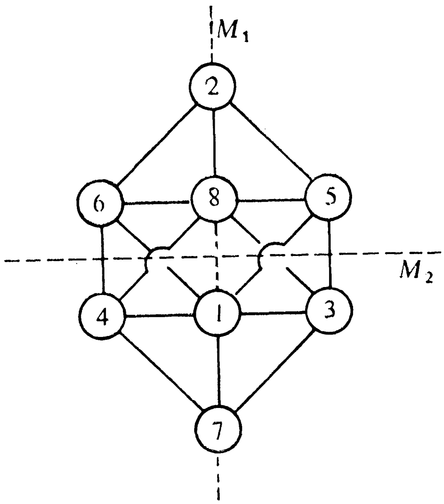

DIAGRAM 2

These are the solutions obtained from the solution in Diagram 2 by reflecting cells across the two orthogonal lines of symmetry M_1 and M_2, rotating by 180 degrees and turning the numbers right side up. To show that these are the only possible solutions, we need only show that we arrive at a contradiction if the 1 and 8 are not in the two six-cells C and F. Suppose 1 is placed in cell B. Then 2 must be in one of H, G, or D. Then we are faced with the problem of putting six consecutive numbers in a no-touch relationship in one of the following cell systems:

Diagram 3 is clearly impossible since cell C is related to all of the other five cells.

A simple counting argument shows that it is impossible to put six consecutive numbers in a no-touch relationship in Diagram 4 since this would imply the absurdity that four consecutive numbers are in a no-touch relationship in a four-element cell system in which every cell is related to two others.

Now we show that Diagram 5 is also absurd. Cells C and F must contain the end numbers in the sequence since they are related to four other cells and if a nonend number were placed in one of these cells, a four element subset of the five remaining numbers would contain a number adjacent to the interior number. But, by hypothesis, G contains 2 and is adjacent to both C and F. Either cell C or F must have the 3 in it, which yields a contradiction.

Finally, to eliminate the other ridiculous possibility, suppose an end number were in one of the two 3-cells. Say 1 were in cell A. Then we would be faced with the problem of putting seven consecutive numbers in a no-touch relationship in a seven element cell system one of whose members has six relatives.

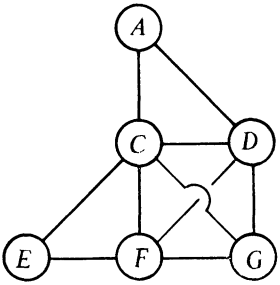

DIAGRAM 3

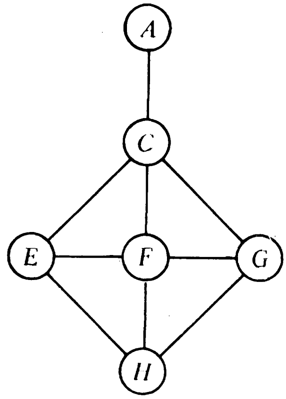

DIAGRAM 4

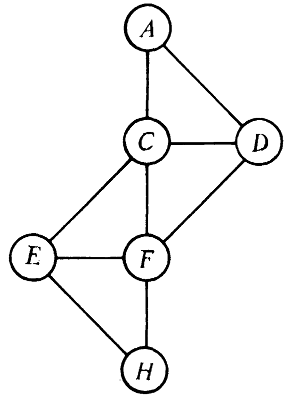

DIAGRAM 5

Thus, the set of all solutions of this no-touch puzzle is invariant with respect to the group of transformations generated by reflections across two perpendicular mirror planes, which is well known to be Klein's four-group. This shows that if you suppose 1 and 8 are adjacent, the puzzle is impossible.

Now let us generalize the problem.

DEFINITION 1. *We say that R is an irreflexive relation on S if R is a subset of $S \times S$, the set of all ordered pairs of elements of S, such that (s,s) is not a member of R for any s in S.* (See [1], page 7.) *The pair (S,R) is called a digraph. If $(s,t) \in R$ we say t and s are R-relatives of each other.*

Let S be an n-element set equipped with an irreflexive relation R and let f be a one-to-one mapping of $\{1,2,\cdots,n\}$ onto S. Let g be the inverse of f. We say that the assignment of integers $\{1,2,\cdots,n\}$ to the cells of S by the rule f satisfies the no-touch condition if whenever (s,t) belongs to the relation R on S, then $g(s)$ and $g(t)$ are not adjacent integers.

The following series of definitions provides us with a further generalization. In the rest of the paper we will generalize the techniques used in finding all of the solutions of the original no-touch problem and will show that for every finite group G there is a generalized no-touch problem such that G is isomorphic to the group of transformations under which the set of all solutions is invariant.

DEFINITION 2. *A relation A on a set I is said to be an adjacency relation if it is symmetric in the sense that (m,n) belongs to A implies (n,m) belongs to A and is irreflexive in the sense of Definition 1. A set I equipped with an adjacency relation A is said to be a graph* (Harary [3]). (*A graph is a special case of a digraph.*)

DEFINITION 3. *Let I and S be sets with the same cardinalities. Suppose (I,A) is a graph and (S,R) is a digraph. A mapping $g: S \to I$ is a bijection if g is one-to-one and onto. We say that a bijection $g: S \to I$ solves the no-touch problem if $(s,t) \in R$ implies $(g(s), g(t)) \notin A$. If no such bijection exists we say that the no-touch problem defined by the four-tuple $(S,R;\, I,A)$ is unsolvable.*

The following are generalizations of theorems used in solving the initial no-touch problem.

THEOREM 1. *Let I and S be p-element sets. Suppose (I,A) is a graph* (Definition 2) *such that one element n of I has at least two A-relatives. Suppose (S,R) is a digraph* (Definition 1) *such that each s in S has at least $p-2$ R-relatives. Then there is no bijection $g\colon S \to I$ which solves the no-touch problem* (Definition 3).

Proof. Let g be any bijection: $S \to I$. Let s be that element of S such that $g(s) = n$. Let $\{s_1, s_2, \cdots, s_{p-2}\}$ be a set of R-relatives of s. Suppose $\{m_1, m_2\}$ contains two A-relatives of n. Since $\{m_1, m_2\} \cap \{g(s_1), g(s_2), \cdots, g(s_{p-2})\} \neq \emptyset$, there is an $i \in \{1,2\}$ and a $j \in \{1, 2, \cdots, p-2\}$ such that $g(s_j) = m_i$. But $(s, s_j) \in R$ or $(s_j, s) \in R$, and both $(n, m_i) = (g(s), g(s_j))$ and $(m_i, n) = (g(s_j), g(s))$ are in A. Thus, the no-touch problem is unsolvable.

THEOREM 2. *Let I and S be $(p+2)$-element sets. Suppose (I,A) is a graph* (Definition 2), *and that I contains two elements m and n satisfying the condition that there exist four elements $m_1, m_2, n_1,$ and n_2 in I such that $\{(m,m_1),(m,m_2),(n,n_1),(n,n_2)\} \subset A$, $m \notin \{n_1, n_2\}$, and $n \notin \{m_1, m_2\}$. Let (S,R) be a digraph* (Definition 1) *and suppose there is an s_0 in S such that the set T of R-relatives of s_0 has at least p elements and such that each t in T is R-related to $p-2$ other elements of T. Then the no-touch problem defined by $(S,R;I,A)$ is unsolvable* (Definition 3).

Proof. Let $g\colon S \to I$ be an arbitrary bijection. If $g(s_0)$ had two A-relatives, then one of those relatives would be in $g(T) = \{g(t)\colon t \in T\}$, and consequently $g\colon S \to I$ would not solve the no-touch problem. Thus, assume $g(s_0)$ has only one A-relative, $g(s_1)$. Write $S = T \cup \{s_0, s_1\}$. Let $\tilde{R} = (T \times T) \cap R$ and let $\tilde{A} = (g(T) \times g(T)) \cap A$. Let m_0 be the A-relative of $g(s_0)$. Then $m_0 \notin g(T)$. Let $\{m_1, m_2\}$ be relatives of m and $\{n_1, n_2\}$ be relatives of n so that $\{m_1, m_2\} \cap \{n_1, n_2\} = \varnothing$. If $m_0 \neq m$ and $m_0 \neq n$ then either $m_0 \notin \{m, m_1, m_2\}$ or $m_0 \notin \{n, m_1, m_2\}$, either of which would imply that the no-touch problem for $(T, \tilde{R};\, g(T), \tilde{A})$ is not solvable, in view of Theorem 1. Suppose therefore that $m_0 = m$. Then $m \notin \{n_1, n_2\}$ implies that $\{n, n_1, n_2\} \subset g(T)$. Thus, Theorem 1 implies the no-touch problem for $(T, \tilde{R};\, g(T), \tilde{A})$ is not solvable. But since g is an arbitrary bijection, it follows that the problem defined by $(S,R;I,A)$ is unsolvable, in view of the definition of $\tilde{R}$ and $\tilde{A}$.

THEOREM 3. *Let S and I be $(p+2)$-element sets. Suppose (I,A) is a graph* (Definition 2). *Suppose (S,R) is a digraph and that there is an s in S which is R-related to at least q elements of S. If a bijection $g\colon S \to I$ solves the no-touch problem for the four-tuple $(S,R;I,A)$ then $g(s)$ can have at most $p+1-q$ A-relatives.*

Proof. Suppose $g(s)$ had as many as $p+2-q$ relatives $\{m_1\ m_2\ \cdots\ m_{p+2-q}\}$. Let $s_1, s_2, \cdots, s_q$ be the R-relatives of s. Then if g satisfied the no-touch condition, it would follow that $\{g(s)\} \cap \{g(s_1), \cdots, g(s_q)\}$ is empty and $\{m_1, \cdots, m_{p+2-q}\}$

$\cap \{g(s_1), \cdots, g(s_q)\}$ is empty since if $m_i = g(s_j)$, the fact that m_i is A-related to $g(s)$, would imply that the no-touch condition is violated. Thus, $\{g(s)\} \cup \{m_1, \cdots, m_{p+2-q}\} \cup \{g(s_1), \cdots, g(s_q)\}$ would be a disjoint union, which is absurd since I contains only $p+2$ elements.

The Symmetry Group of the No-Touch Problem. Let $(S,R;I,A)$ be a four-tuple of sets and relations satisfying the conditions of Definition 3. Suppose there is a bijection $g: S \to I$ such that $(s,t) \in R$ implies $(g(s), g(t)) \notin A$. We say that a bijection $\sigma: S \to S$ is *solution-preserving* if the composition $g \circ \sigma: S \to I$ is a solution of the no-touch problem in the sense of Definition 3. The set $G(S,R;I,A)$ of all permutations σ of S, such that $g \circ \sigma: S \to I$ is a solution of the no-touch problem whenever $g: S \to I$ is a solution of the no-touch problem, is clearly a group, which we call the symmetry group of the no-touch problem. In the case of the original puzzle, this group was Klein's four group.

For every finite group G, there is a digraph (S,R) and a graph (I,A) such that the symmetry group of the no-touch problem defined by the four-tuple $(S,R;I,A)$ is isomorphic to G. This is a consequence of the Frucht construction ([**2**], [**3**]), which shows that there is a graph (S,R) whose group $\Gamma(S)$ of permutations of S, which transform R-related elements of S into related elements, is isomorphic to the given finite group G. This theorem answered a question posed by König [**4**]. Let $I = S$ and let A be the complement in $S \times S$ of R. Then it can be shown that $G(S,R;I,A)$, the symmetry group of the no-touch problem defined by $(S,R;I,A)$, is equal to $\Gamma(S)$ which is isomorphic to G.

Acknowledgment. I wish to thank the referee for his helpful comments.

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NORMALIZING VOGEL'S APPROXIMATION METHOD

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This paper describes an approximation method for determining an initial basic feasible solution to transportation type problems. The paper presents a "cost normalization" which is easily performed with the aid of a computer. In over 200 random problems tested, this normalizing (while using little time itself) has reduced the average number of iterations required to reach an optimal solution to a third of that required after using Vogel's approximation as an initial solution.

This paper is a result of an independent study group connected with a freshman linear algebra course given by the author during the spring of 1971 at the Behrend Campus of Pennsylvania State University. The naiveté of the approach seems to be in refreshing contrast to the significant refinement of the approximation. I would like to give particular credit to two of the eight students in the study group: to John Pietron for the genesis of the idea, and to James Martin for his aid in designing a computer program to test the method.

The approximation we will describe is, of course, independent of any iterative procedure for refinement to an optimal solution. However, when speaking of the number of iterations required to reach an optimal solution, we refer to the stepping-stone method described in Hadley's *Linear Programming* [**1**].

The body of this paper is divided into three sections. In the first section, we describe the normalizing procedure. In the second, we give an example of a problem which illustrates both the approximation and its accuracy. Finally, in the third section we describe the results of testing the accuracy of the approximation.

A general transportation problem may be stated as follows. We wish to find a solution matrix $[x_{i,j}]$ which minimizes

$$z = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{i,j}\, x_{i,j} \text{ given the restraints}$$

$$x_{i,j} \geqq 0,$$

$$\sum_{i=1}^{n} x_{i,j} = a_j,\ a_j > 0, \quad j = 1, 2, \cdots, m,$$

$$\sum_{j=1}^{m} x_{i,j} = b_i,\ b_i > 0, \quad i = 1, 2, \cdots, n, \text{ and}$$

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j.$$

There are several different methods one can use to determine an initial basic feasible solution. For example, see [**1**]. As we began this study, the method which seemed to us to have achieved the nicest balance between simplicity and accuracy was Vogel's approximation method. For a description of this method, see [**1**] [**2**] [**3**].

Before proceeding with section one, we would like to point out that the normalizing process we have incorporated into Vogel's method is not recommended for hand calculations, but may be carried out quickly by use of a computer.

I. Cost normalization. The normalized approximation may be described as follows. Rather than comparing the actual costs given in the cost matrix $[c_{i,j}]$, to determine an initial basic feasible solution, we replace each $c_{i,j}$ by a normalized "cost" and compare these. We define the normalized cost

$$C_{i,j} = c_{i,j} - \frac{1}{n} \sum_{p=1}^{n} c_{p,j} - \frac{1}{m} \sum_{q=1}^{m} c_{i,q} .$$

In other words, from each cost we simply subtract the average cost of the row and column in which it appears. Once this has been done, we apply Vogel's approximation to this normalized cost matrix $[C_{i,j}]$.

Although we will only use this normalized cost to obtain an initial solution, we point out in the following theorem that both cost matrices yield the same optimal solutions.

THEOREM. *A solution $[x_{i,j}]$ to a transportation problem is optimal with respect to a cost matrix $[c_{i,j}]$ iff it is optimal with respect to the corresponding normalized cost matrix $[C_{i,j}]$.*

Proof. Let $z = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{i,j} x_{i,j}$ and $Z = \sum_{i=1}^{n} \sum_{j=1}^{m} C_{i,j} x_{i,j}$. Also, let $A = \sum_{j=1}^{m} \sum_{p=1}^{n} c_{p,j} a_j$ and $B = \sum_{i=1}^{n} \sum_{q=1}^{m} c_{i,q} b_i$.

$$Z = \sum_{i=1}^{n} \sum_{j=1}^{m} \left(c_{i,j} - \frac{1}{n} \sum_{p=1}^{n} c_{p,j} - \frac{1}{m} \sum_{q=1}^{m} c_{i,q} \right) x_{i,j}.$$

By reordering the above sum, we see that

$$Z = \left[\sum_{i=1}^{n} \sum_{j=1}^{m} c_{i,j} x_{i,j} \right] - \left[\frac{1}{n} \sum_{j=1}^{m} \sum_{p=1}^{n} c_{p,j} \sum_{i=1}^{n} x_{i,j} \right] - \left[\frac{1}{m} \sum_{i=1}^{n} \sum_{q=1}^{m} c_{i,q} \sum_{j=1}^{n} x_{i,j} \right]$$
$$= z - \left[\frac{1}{n} \sum_{j=1}^{m} \sum_{p=1}^{n} c_{p,j} a_j \right] - \left[\frac{1}{m} \sum_{i=1}^{n} \sum_{q=1}^{m} c_{i,q} b_i \right]$$
$$= z - \left[\frac{1}{n} A + \frac{1}{m} B \right].$$

Since A and B are independent of the solution $[x_{i,j}]$, it is clear that Z is optimal iff z is optimal, and the proof is complete.

If one entire row or column of the cost matrix has zero entries, something which may easily occur in order to satisfy the restriction that $\sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j$, we make the following modification in defining $C_{i,j}$. Say $c_{i,j} = 0$, $i = 1, \cdots, n$. Then define

$$C_{i,j} = c_{i,j} - \frac{1}{n} \sum_{p=1}^{n} c_{p,j} - \frac{1}{m-1} \sum_{q=1}^{m} c_{i,q}.$$

It is easily verified that the above theorem applies to this form of $[C_{i,j}]$ also.

II. Example.

Cost Matrix

	90	87	94	*82*	*75*	80	30	
	230	*220*	235	240	223	*227*	40	
$[c_{i,j}]$	175	182	*178*	190	180	*176*	30	$[a_i]$
	136	134	140	*138*	143	150	10	
	55	58	*60*	*61*	57	64	30	
$[b_j]$	11	13	22	31	21	42		

Normalized Cost Matrix

	−131.8	−133.8	−132.0	*−144.8*	*−145.2*	−144.0	30	
	−136.3	*−145.3*	−135.5	−131.3	−141.7	*−141.5*	40	
$[C_{i,j}]$	−142.3	−134.3	*−143.5*	−132.3	−135.7	*−143.5*	30	$[a_i]$
	−141.3	−142.3	−141.5	*−144.3*	−132.7	−129.5	10	
	−141.3	−137.3	*−140.5*	*−140.3*	−137.7	−134.5	30	
$[b_j]$	11	13	22	31	21	42		

Even though our testing was conducted primarily on larger cost matrices than the above, we use this 5×6 matrix for simplicity and lack of space.

We have italicized the costs which enter into the optimal solution in each of the above matrices. Note that the "costs" in the second matrix are negative.

Vogel's approximation for $[c_{i,j}]$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 30 \\ 0 & 13 & 0 & 6 & 21 & 0 \\ 0 & 0 & 3 & 15 & 0 & 12 \\ 0 & 0 & 0 & 10 & 0 & 0 \\ 11 & 0 & 19 & 0 & 0 & 0 \end{bmatrix}$$

$z = 20{,}004$

Vogel's approximation for $[C_{i,j}]$

$$\begin{bmatrix} 0 & 0 & 0 & 9 & 21 & 0 \\ 0 & 13 & 0 & 0 & 0 & 27 \\ 0 & 0 & 15 & 0 & 0 & 15 \\ 0 & 0 & 0 & 10 & 0 & 0 \\ 11 & 0 & 7 & 12 & 0 & 0 \end{bmatrix}$$

$z = 19{,}749$

Vogel's approximation requires three iterations to reach an optimal solution, while the normalized Vogel's approximation is optimal.

III. The preceding example in section two was chosen to be representative of our test results. These results have been summarized in the following table:

Dimensions of the Cost Matrix	*Number of Matrices Tested*	I_1	I_2
10 × 10	150	11	4
10 × 20	50	22	7
25 × 25	10	59	23

I_1 is the average number of iterations required by Vogel's approximation to reach optimality. I_2 is the average number of iterations required by the normalized Vogel's approximation.

Of the matrices randomly chosen in our computer program, the number of iterations required by Vogel's approximation varied greatly with each cost matrix, while the number of iterations required by the normalized Vogel's approximation was much more consistent. For example, one of the 10 × 20 matrices required 38 iterations after Vogel's approximation, while the greatest number of iterations required by any 10 × 20 matrix after the normalized approximation was 9.

In conclusion, we point out that none of the randomly chosen matrices required a greater number of iterations for the normalized approximation than for Vogel's approximation.

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THE TRIANGLE AREA FORMULA IMPLIES THE PARALLEL POSTULATE

C. K. SHENE, Tamkang College of Arts and Sciences, Taipei

In his book *Elementary Geometry from an Advanced Standpoint*, Edwin E. Moise gives a proof that the formula "$area(\Delta ABC) = 2^{-1} \cdot h_a \cdot a$" implies the Euclidean parallel postulate. Here h_a is the altitude on side $a(=\overline{BC})$ of ΔABC (cf. pp. 343–347). Here we offer a proof which does not use the Bolyai theorem. We use the notation of Moise's book. Our proof begins with two lemmas.

LEMMA I. *Let* $\overleftrightarrow{AD}$, $\overleftrightarrow{BC}$ *be straight lines with* $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$ *both perpendicular to to* $\overleftrightarrow{BC}$. *If* $\overline{AB} \cong \overline{CD}$, *then*

(i) $\angle BAD \cong \angle CDA$

(ii) $\overleftrightarrow{AD} \cap \overleftrightarrow{BC} = \varnothing$ (*the empty set*).

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(i) $\angle BAD \cong \angle CDA$

(ii) $\overleftrightarrow{AD} \cap \overleftrightarrow{BC} = \varnothing$ *(the empty set).*

Proof. Let M, N be the midpoint of $\overline{AD}$, $\overline{BC}$ respectively (see Figure 1). We see easily that $\Delta ABN \cong \Delta DCN$ (S.A.S.) and then $\Delta AMN \cong \Delta DMN$ (S.S.S.). By these two relations we have

(1) $m\angle ANB = m\angle DNC$

(2) $m\angle ANM = m\angle DNM$

(3) $m\angle BAN = m\angle CDN$

(4) $m\angle NAM = m\angle NDM$

(5) $m\angle AMN = m\angle DMN = 90.$

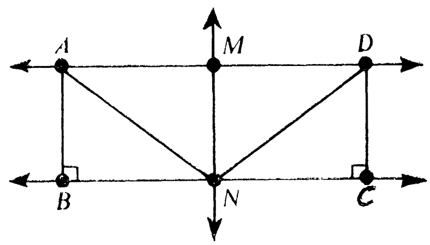

FIG. 1

Adding (3), (4) we have (i). Adding (1), (2) and combining (5) it follows that $\overleftrightarrow{MN}$ is the common perpendicular of $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$. Therefore $\overleftrightarrow{AD}$, $\overleftrightarrow{BC}$ do not intersect.

LEMMA II. *In hyperbolic geometry, for every triangle ΔABC (resp. quadrilateral $\square ABCD$) we have $m\angle A + m\angle B + m\angle C < 180$ (resp. $m\angle A + m\angle B + m\angle C + m\angle D < 360$).*

From now on we assume that the formula

$$(\#) \qquad area(\Delta ABC) = 2^{-1} \cdot h_a \cdot a$$

is true and prove the Euclidean parallel postulate.

EXISTENCE THEOREM. *If P is a given point not on a given line $\overleftrightarrow{AC}$, then there is a line passing through P and not intersecting $\overleftrightarrow{AC}$.*

Proof. Choose a point B on $\overleftrightarrow{AP}$ such that $A - P - B$ and $\overline{AP} \cong \overline{PB}$. Construct a line passing through P and the midpoint Q of $\overline{BC}$. We claim that $\overleftrightarrow{PQ}$ has the desired property. For, we have (see Figure 2)

$$area(\Delta APC) = 2^{-1} \cdot area(\Delta ABC) \qquad \text{(by } (\#))$$

$$= area(\Delta AQC) \qquad \text{(by } (\#))$$

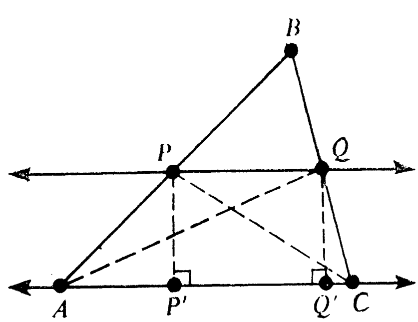

FIG. 2

But ΔAPC, ΔAQC have common base $\overline{AC}$, then we have $\overline{PP'} \cong \overline{QQ'}$ by using (#) again, where $\overline{PP'}$, $\overline{QQ'}$ are the altitudes of ΔAPC, ΔAQC respectively. Now our theorem follows from Lemma I.(ii).

UNIQUENESS THEOREM. *The line referred to in the Existence Theorem is unique.*

Proof. Our method is by contradiction. We choose C so that $\overline{AB} \cong \overline{BC}$ (see Figure 3). From A, B, C draw perpendiculars to $\overleftrightarrow{PQ}$ intersecting $\overleftrightarrow{PQ}$ in L, M, N respectively. Let D be a point on the same side of $\overleftrightarrow{PQ}$ as B such that the distances from B, D to $\overleftrightarrow{PQ}$ are equal (i.e., $\overline{BM} \cong \overline{DY}$ in Figure 3). It is not difficult to see that

$$\overline{AL} \cong \overline{BM} \cong \overline{DY} \cong \overline{CN}$$

and

$$\begin{aligned} area(\Delta ABC) &= area(\square LACN) \\ &= area(\Delta ADC). \end{aligned}$$

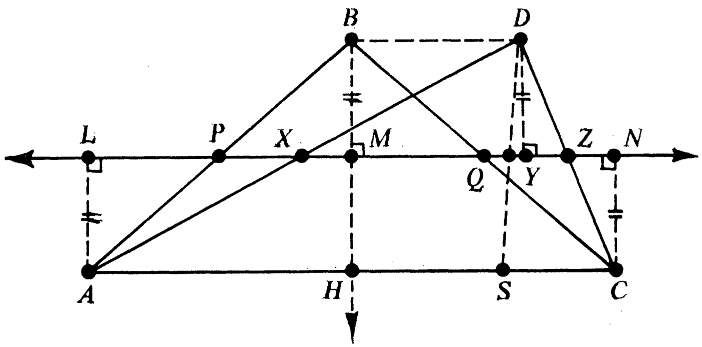

FIG. 3

If the theorem is false, then there are at least two lines passing through P and not intersecting $\overleftrightarrow{AC}$. Clearly, this is just the system of hyperbolic geometry.

Let $\overrightarrow{BM}$ intersect $\overline{AC}$ in H. From D draw a perpendicular to $\overleftrightarrow{AC}$ intersecting $\overleftrightarrow{AC}$ in S.

Assertion I. $\overleftrightarrow{BH} \perp \overleftrightarrow{AC}$.

From the construction above we know that

$$BP = 2^{-1} \cdot AB = 2^{-1} \cdot BC = BQ$$

$$BM = BM$$

$$m\angle BMP = m\angle BMQ = 90.$$

Therefore

$$\Delta BPM \cong \Delta BQM$$

and hence

$$\angle ABH \cong \angle CBH.$$

Applying the last fact and the S.A.S. law it follows that $\Delta ABH \cong \Delta CBH$ and hence $\angle AHB \cong \angle CHB$. Thus $\overleftrightarrow{BH} \perp \overleftrightarrow{AC}$.

Assertion II. D, Y, S are not collinear.

If Y is on $\overleftrightarrow{DS}$, then we have a quadrilateral $\square MHSY$ such that

$$\begin{aligned} &m\angle HMY + m\angle MYS + m\angle YSH + m\angle SHM \\ &= 4 \times 90 \\ &= 360. \end{aligned}$$

This contradicts Lemma II.

Assertion III. $\overline{BH}$, $\overline{DS}$ have different length.

If they have the same length, then Lemma I.(i) gives us $\angle YDB \cong \angle MBD \cong \angle SDB$. Because Y, S are on the same side of $\overleftrightarrow{AD}$, it follows $\overleftrightarrow{DY}$, $\overleftrightarrow{DS}$ coincide. This contradicts Assertion II.

Therefore we have constructed two triangles ΔABC, ΔADC with the same base $\overline{AC}$ and area ($area(\square LACN)$) but different altitude. Of course this contradicts the formula (#) and hence our proof is complete.

BOOK REVIEWS

EDITED BY D. ELIZABETH KENNEDY, University of Victoria

Materials intended for review should be sent to Professor D. Elizabeth Kennedy, Department of Mathematics, University of Victoria, Victoria, British Columbia, Canada.

Reviews of texts at the freshman-sophomore level based upon classroom experience will be welcomed by the Book Review Editor.

A boldface capital **C** *in the margin indicates a classroom review.*

Topics from Triangle Geometry. By D. Moody Bailey. Privately printed, Princeton, West Virginia, 1972. iv + 258 pp.

Though this book is not adaptable as a text for any of the usual courses in so-called modern, or college, geometry, and though the bulk of the material of the book is rather disconnected from most current work in this area, the instructor and/or devotee might like a copy for his personal library. The book is made up of some sixteen essentially independent papers, a number of which are associated with some earlier papers by the author that have appeared in *School Science and Mathe-*

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matics and this MAGAZINE. Most of the papers, by long and laborious computation, arrive at complicated expressions for ratios involving certain line segments of a triangle, or complicated expressions for the powers of certain points of a triangle with respect to certain circles of the triangle. Basic to most of these relations are the famous theorems of Ceva and Menelaus. The book is neatly printed, well proof-read, and possesses a number of good line diagrams.

HOWARD W. EVES, University of Maine, Orono

WHO READS THE MATHEMATICS MAGAZINE?

L. J. COTE and R. P. O'MALLEY, Purdue University

A survey was recently undertaken by the editors of the MATHEMATICS MAGAZINE to find out something about the MAGAZINE'S reading audience. The following is a report of the results of that survey.

In February, 1972, a questionnaire was mailed to a sample of subscribers to get their reactions to various aspects of the MAGAZINE. This questionnaire is reproduced on pages 274–75.

Selecting the sample for the survey produced some interesting facts about the readers even before the questionnaire was mailed out. The systematic sample was chosen by taking every 15th address from the MM's subscription list of slightly over 7000. About 44% of the subscribers turned out to be libraries or other institutions. Elimination of these as possible respondents to the survey and elimination of all foreign subscribers due to impracticality in terms of time and expense, left 244 to whom the questionnaire was sent. This was 52% of the total sample.

By the deadline, March 15th, a total of 108 responses had been received. This was approximately a 44% response, a respectable percentage for a mailed questionnaire. In the interpretation of the results the possibility that the nonrespondents differ from the respondents in the qualities which the survey is trying to measure must be considered. A more careful survey would have provided for an estimate within the nonresponding group. This survey, however, is the first in the MM's experience, with the questions and sampling method both untested, so it was decided to avoid this complication. Ignoring this bias, the following facts were found about the MM's readers.

Table I gives the demographic pattern given by the response to Question $\neq$ 1. It is worthy of remark that the nonacademic subscribers form the third largest group. Also, surprisingly few students are readers of the MAGAZINE. Of the 11 students, three are in undergraduate colleges, 8 are in universities with a Ph. D. program in math, and none are in high schools or junior colleges. The group of teachers in undergraduate colleges or universities that do not grant a Ph. D. in math is significantly larger than the others. However, categories 1, 3, and 5 do not differ significantly from one another (from a total of 108, as a rough figure, a significant difference between groups would be more than 8 persons).

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QUESTIONNAIRE

This questionnaire is being sent to a small, randomly chosen sample of about 250 subscribers to MATHEMATICS MAGAZINE. We would appreciate your taking the time to complete it and return it to us not later than March 15, 1972, in the enclosed stamped, self-addressed return envelope. A report on the poll will be published in the MATHEMATICS MAGAZINE.

1. I am a { □ teacher / □ student } in a { □ secondary or lower level school / □ two-year college / □ undergraduate college / □ university granting the Ph.D. in math / □ university not granting the Ph.D. in math }

   I am □ none of the above

2. The problem section { □ does / □ does not } interest me.

3. I { □ never / □ sometimes / □ usually / □ always } read the book reviews.

4. Please list the 3 articles in the 1971 issues of Mathematics Magazine which were of most interest to you.
   1.
   2.
   3.

5. For each of the articles listed below, please put in the brackets the number 0, 1, 2 or 3 according to the following code.

   [0] I did little or no more than read the title
   [1] I skimmed through it only
   [2] I skimmed through it and read part of it in detail
   [3] I read it in detail

Daykin, The Bicycle Problem, Jan 72	[ ]
Darst, Some Cantor Sets and Cantor Functions, Jan. 72	[ ]
Dolan, Early Sundials and the Discovery of the Conic Sections, Jan. 72	[ ]
Brenner and DePillis, Fermat's Equation for Matrices of Integers, Jan. 72	[ ]
Barr, When Will the Next Record Rainfall Occur? Jan. 72	[ ]
Jones and Landau, One-Sided Limits and Integrability, Jan. 72	[ ]
Hagis, Relatively Prime Amicable Numbers with 21 Prime Divisors, Jan. 72	[ ]
Ash and Bishop, Monopoly as a Markov Process, Jan. 72	[ ]
Federico, The Melancholy Octahedron, Jan. 72	[ ]
Usiskin and Wayment, Partitioning a Triangle into 5 Triangles Similar to It, Jan. 72	[ ]
Carlitz, Some Inequalities for Two Triangles, Jan. 72	[ ]
Milner, A Note on the Graphs of Groups, I, Jan. 72	[ ]
Goffman, A Note on Integration, Jan. 71	[ ]
Deans, Covariant and Contravariant Vectors, Jan. 71	[ ]
Stanley, A Note on the Sequence of Fibonacci Numbers, Jan. 71	[ ]
Chakerian and Lange, Geometric Extremum Problems, March 71	[ ]
Giudici, Quadratic Residues in $GF(p^2)$, May 71	[ ]
Sell, Completion of a Metric Space, Sept. 71	[ ]
Liebeck, Some Generalizations of the 14–15 Puzzle, Sept. 71	[ ]
Jordan and Schneider, Covering Classes of Residues in $Z(\sqrt{-2})$, Nov. 71	[ ]

Please list here or on the back of these pages any comments which you have on any aspect of the Mathematics Magazine.

TABLE 1.

Category		# of respondents	% of total
1	Teacher in secondary or lower level school or 2-year college	20	18.5%
2	Teacher in undergraduate college or university not granting the Ph.D. in math	42	39.0%
3	Teacher in university granting the Ph.D. in math	16	14.8%
4	Student	11	10.1%
5	None of the above	19	17.6%

The responses to Questions # 2 and # 3 of the questionnaire speak well for the MAGAZINE. Table 2 shows that about 90 per cent of the readers in each category find the Problem Section interesting, and one third of the teachers, who make up 72% of the sample, are constant readers of the book reviews.

TABLE 2.

Category	Interested in Problem Section	Read Book Reviews			
		Never	Sometimes	Usually	Always
1	95%	0%	25%	40%	35%
2	88%	0%	33%	31%	36%
3	75%	0%	25%	37.5%	37.5%
4	99%	36%	36%	18%	10%
5	90%	10%	53%	16%	21%

The articles listed as favorites by the respondents in Question # 4 included a large proportion of the articles published in 1971. In 1971 there were 76 articles published in the MM, and 55 of these were listed at least once by a respondent. The most popular articles, judging from the response to Question # 4, seem to be those of a nontechnical nature. Articles on games like instant insanity seemed to be very popular with readers.

The results of Question # 5 are given in Table 3. The grades of 0 to 3 which were given to the listed articles were treated as quantities and averaged. The articles are listed in order of highest average to lowest. It is interesting to note the titles in this list — the most popular articles have the least technical titles. Perhaps this is one significant result of this survey. Readers of the MM may be "put off" by the titles of some articles. It would seem that a title which is indicative, in non-technical language, of the content of the paper attracts the largest audience for that paper.

TABLE 3.

Author	Title	Averages Total	Categories 1	2	5
Daykin	The Bicycle Problem	1.49	1.44	1.40	1.58
Dolan	Early Sundials and the Discovery of the Conic Sections	1.37	1.66	1.26	1.37
Barr	When Will the Next Record Rainfall Occur?	1.35	1.47	1.33	1.67
Stanley	A Note on the Sequence of Fibonacci Numbers	1.30	1.56	1.31	1.27
Ash	Monopoly as a Markov Process	1.28	1.06	1.07	1.84
Goffman	A Note on Integration	1.21	1.37	1.31	.86
Darst	Some Cantor Sets and Cantor Functions	1.19	1.00	1.23	1.11
Sell	Completion of a Metric Space	1.12	.94	1.18	.86
Jones	One-Sided Limits and Integrability	1.09	1.16	1.05	1.00
Federico	The Melancholy Octahedron	1.07	1.12	.95	1.21
Brenner	Fermat's Equation for Matrices of Integers	1.02	1.16	1.12	1.11
Usiskin	Partitioning a Triangle into 5 Triangles Similar to It.	1.02	1.33	.88	1.16
Chakerian	Geometric Extremum Problems	1.01	.88	.94	.93
Milner	A Note on the Graphs of Groups	.93	1.05	.92	.67
Hagis	Relatively Prime Amicable Numbers with 21 Prime Divisors	.92	1.11	.85	1.10
Liebeck	Some Generalizations of the 14–15 Puzzle	.85	.80	.77	.93
Deans	Covariant and Contravariant Vectors	.80	.33	.89	.86
Carlitz	Some Inequalities for Two Triangles	.68	.94	.72	.61
Jordan	Covering Classes of Residues in Z $(\sqrt{-2})$	.64	.50	.72	.66
Giudici	Quadratic Residues in $GF(p^2)$	.46	.40	.51	.64

Respondents were asked to make comments on the back of their questionnaires. The comments were mostly favorable, encouraging the status quo. Of the 108, 40 made comments. The most frequent (14) comment was that the respondent liked the MM as it is. Several responses were constructive comments and suggestions for changes. Five respondents used the MM frequently in their classroom work. Five respondents said that they suspected that they should not be subscribers at all, showing that the nonreaders were at least partly represented in the sample.

Who reads MATHEMATICS MAGAZINE? For the most part, teachers in universities or colleges not granting the Ph. D. in mathematics, who read and enjoy the Problem Section regularly, who are interested in the book reviews, and who probably tend to choose which articles they will read by scanning the list of titles on the cover to choose those they think they will find most interesting. The other two groups of teachers seem to be quite similar to this group. The surprisingly large group of nonacademic subscribers is also similar to the teachers, but suggestions of differences appear in the data. The survey results are interesting, and hopefully, useful to the editors.

NOTES AND COMMENTS

Joseph S. Madachy comments as follows on *Integers and the sum of the factorials of their digits* by George D. Poole, November 1971. "The results (1, 2, 145, and 40585) have been known since 1964. The first three were known for many years before, and the last result (40585) was found by Leigh Janes in 1964 using the exhaustive technique on a computer at Davidson College, North Carolina. He used the same technique suggested by Poole and established that there were no others. The results were reported by me in *Mathematics on Vacation* (Scribner's Sons, New York, 1966), page 167. Janes' results were independently checked by Ron S. Dougherty also at the same college. Dougherty went ahead and found the only solution to the problem involving subfactorials ($!0 = 0$, $!1 = 0$. $!2 = 1$, $!3 = 2$, $!4 = 9$, $!5 = 44$, $!6 = 265$, $!7 = 1854$, $!8 = 14833$, $!9 = 133496$):

$$148349 = !1 + !4 + !8 + !3 + !4 + !9.$$

(There are two solutions if the trivial $0 = !0$ is included.)

This was also reported on the same page of the reference (*Math. on Vac.*) given above. In connection with this factorial problem the exhaustive search via computer need not involve all the integers to 2,000,000. In a method which will appear (in 1973) in the Fibonacci Quarterly, it can be shown that only 24310 integers need be examined in order to establish the complete search for these numbers. It was by using this technique that Patton, Patton, and I were able to find such seemingly impossibly elusive integers as:

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$$233411150132317 = 2^{17} + 3^{17} + 3^{17} + 4^{17} + 1^{17} + 1^{17} + 1^{17} + 5^{17} + 0^{17} + 1^{17} + 3^{17} + 2^{17} + 3^{17} + 1^{17} + 7^{17}.$$

This did *not* involve searching something like 10^{15} integers—only 4,686,825 (or less) were searched. *All* such integers involving the 17th powers (there are 3 more) are now known. Unfortunately the technique cannot be used to find such integers as

$$47016 = 4^2 + 7^3 + 0^4 + 1^5 + 6^6$$

(exponents increase arithmetically). This, and similar results, appear in *Some new narcissistic numbers*, The Fibonacci Quarterly, Vol. 10, No. 3, April 1972, pp. 295–298."

Paul J. Campbell, St. Olaf College, writes regarding *Some extensions of nim* by Benjamin L. Schwartz in the November 1971 issue that it gives "a very well-written exposition of the winning strategy for the game of 'bounded nim'. Its author believed that the strategy had not been published previously. The same game, however, was described on p. 72 of *Compound games with counters* by Cedric A. B. Smith in the J. Recreational Math. (1968), p. 67–77, where it was called 'Nim(M)'. The winning strategy was also given." References to other papers on the general theory of such games are given there. Professor Campbell also calls attention to an article *Mathematical games* by Martin Gardner in the January 1972 issue of Scientific American.

Notes and Comments in the November 1971 issue included a comment by J. A. H. Hunter on a paper by Makowski in the September 1970 issue on sums of squares of consecutive odd integers. Hunter claimed that in the case $n = 241$ it can easily be shown that there is no solution. Hunter now writes: "A few days ago I had a letter from William Sollfrey giving an actual numerical resolution for $n = 241$. It could not be disputed. That case depended on the solution of

$$X^2 - 241Y^2 = -2 \cdot 5 \cdot 11^2, \text{ where } X = x + 240,$$

$z = 241Y$. (See Brother Alfred's January 1964 paper for definition of x and z.)

I based my so-called 'proof' on the impossibility of $A^2 - 241B^2 = -10$. Wishful thinking, for in fact there could be no valid justification for assuming that as a criterion. Sollfrey found: If $A^2 - 241B^2 = -160$, the minimal solution (to give positive final x) is

$$A = 1042557 \qquad B = 67157$$

leading to $x = 11{,}467{,}887 \quad z = 178{,}033{,}207$.

Now I have seen that the case of $n = 241$ *could have been* shown to have a solution very simply indeed without any undue amount of figuring. To do this it was not in fact necessary to produce a final numerical solution as was produced by Sollfrey. It is so easy to be wise after the event! We have, say,

$$X^2 - 241Y^2 = -2^5 \cdot 5 \cdot 11^2.$$

Successive denominators in the first half period for the continued-fraction expansion of $\sqrt{241}$ are

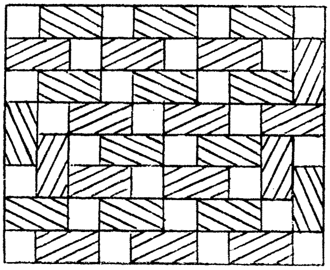
16, 15, 3, 24, 5, 9, 8, 15.

So there can be no solution for $P^2 - 241Q^2 = \pm 11$. Hence we *must* have $X = 11m$, and $Y = 11n$, with $m^2 - 241n^2 = -160$. But, at sight, $9^2 - 241 \cdot 1^2 = -160$. So there is an infinity of solutions for $m^2 - 241n^2 = -160$ whence there will be corresponding solutions for the vital equation $X^2 - 241Y^2 = -2^5 \cdot 5 \cdot 11^2$. THAT IS THE SUFFICIENT PROOF FOR $n = 241$".

P. J. Federico writes regarding his paper, *The melancholy octahedron*, in the Jan. 1972 issue: "The December 1971 number of the German art magazine *Die Kunst*, which was received after *The melancholy octahedron* was printed, contains an article by Curt Seckel with the title *Die Symbolik in Dürer's 'Melancholie'*". It appears from this article that there is in existence a notebook of Dürer containing preliminary or trial sketches, with construction lines, relating to some of his work — the 'Dresdner Skizzenbuch'. This notebook contains a sketch of the stone block with the construction of the rear shown in dotted lines, which sketch is reproduced in Seckel's article. It turns out to be remarkably similar to Figure 2A on page 32 of the January issue of this MAGAZINE, including a closeness in perspective, but in mirror image as the lines would be placed on the copper plate. So there was no mystery after all; the stone block is in fact the octahedron formed by truncating the two opposite vertices farthest removed from each other, of a properly proportioned rhombohedron."

He also points out that in line 12 of the first paragraph on page 30 the words "and a ground line" should be cancelled, as this line was omitted in reproducing the figures.

Barbara Huval sent us this comment on *The gunport problem* by Bill Sands in the September 1971 issue. "This fascinating problem was presented to me by Dr. Berzsenyi as a homework assignment for a Math 131 class. Within 24 hours after I brought the problem home, our whole family was immersed in dominoes, happily arranging and rearranging gunports. Mr. Sands commented that mathematically, it should be possible to construct 26 gunports using 27 dominoes on an 8×10 board, but in actual practice he had never seen this done. May I submit the enclosed solution, (see p. 281) originally conceived by my husband, Capt. John C. Huval."



H. W. Gould, West Virginia University, expresses surprise that Wiggin's paper, *An occupancy problem involving placement of pairs of balls*, March 1972, was published because the proofs given are "complicated and obscure the essential simplicity of the problems. We only need to realize that a unit of m objects may be treated as single X to see that we must subtract mk boxes from the given n boxes and then add in k boxes (each of which accommodates a single X), whence we have precisely $n - mk + k$ boxes to fill with k X's and $n-mk$ non-X's, so that the answer must be the same as the number of ways of seating k persons in a row of $n - mk + k$ seats, or $\binom{n-mk+k}{k}$."

Michael Capobianco, Notre Dame College of St. John's University, Staten Island, also sent us a simpler solution of this problem.

Kumar Jogdeo, University of Illinois at Urbana-Champaign, notes that the inequality proved in the article *Covariance of monotone functions* by Javad Behboodian in the May 1972 issue and claimed there to be new is well known as Tchebyshef's inequality and appears in the book *Inequalities* by Hardy, Littlewood and Polya on pages 43 and 44. He adds that there are several generalizations of this inequality (see for example E. L. Lehmann, Annals of Math. Stat., 37 (1966), pp. 1137–53).

With regard to the paper *Another proof of Tepper's identity*, May 1972, we have received the following comment from a reader. "May I suggest that much valuable time and space has been lost in not observing that the identity

$$r! = \sum_{i=0}^{r} (-1)^i \binom{r}{i} (x-i)^r$$

is a standard exercise in combinatory analysis, and follows from the obvious identity

$$\begin{aligned} f(t) &= \sum_{i=0}^{r} (-1)^i \binom{r}{i} e^{(x-i)t} \\ &= e^{xt}(1-e^{-t})^r \\ &= t^r + \text{higher powers of } t \end{aligned}$$

$$\text{since } f^{(r)}(0) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} (x-i)^r = r!$$

In fact, the method shows considerably more, namely that

$$f^{(s)}(0) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} (x-i)^s = 0$$

when $s < r$."

Similar comments regarding this paper were received from F. J. Papp, University of Lethbridge and another reader who also prefers to remain anonymous.

Regarding the article *A theorem on rational zeros of a polynomial* by Leighton, May 1972, Harley Flanders, Tel Aviv University, writes that if D is a unique factorization domain (U.F.D.) then so is $D[x]$. In particular $\mathbb{Z}[x]$ is a U.F.D. So if $f(x)$ is in $\mathbb{Z}[x]$ and $f(\frac{p}{q}) = 0$ where $(p,q) = 1$ then $f(x) = (x - \frac{p}{q})g(x)$ is in $Q[x]$. Hence $df(x) = (qx-p)h(x)$ where d is in $\mathbb{Z}$ and $h(x)$ is in $\mathbb{Z}[x]$. Then by unique factorization, $d \mid h(x)$ in $\mathbb{Z}[x]$ since $qx-p$ is obviously irreducible. Therefore $u(x) = h(x)/d$ is in $\mathbb{Z}[x]$.

William I. Miller, Kirkland Hall College also sent a simpler proof of Leighton's theorem and Erwin Just, Bronx Community College of CUNY sent two simple proofs of the following converse:

Let $f(x) = \sum_{i=0}^{n} a_i x^i$ in which the a_i are integers. If p and q are relatively prime integers and there exists an infinite set of integers $\{m_i\}$, $i = 1, 2, \cdots$, such that for each i, $(qm_i - p) \mid f(m_i)$, then p/q is a root of $f(x) = 0$.

Interested readers are encouraged to construct for themselves proofs of this converse.

Theorem [Abel]

You may strive all day
At your radical play
But there is no way
To make all quintics say
Uncle.

EDWARD T. HILL

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You may strive all day
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EDWARD T. HILL

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College

ASSOCIATE EDITOR, J. S. FRAME, Michigan State University

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Problems may be submitted from any branch of mathematics and ranging in subject content from that accessible to the talented high school student to problems challenging to the professional mathematician. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

The asterisk (*) will be placed by the problem number to indicate that the proposer did not supply a solution. Readers' solutions are solicited for all problems proposed. Proposers' solutions may not be "best possible" and solutions by others will be given preference.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles Valley College, 5800 Fulton Avenue, Van Nuys, California 91401.

To be considered for publication, solutions should be mailed before May 1, 1973.

PROPOSALS

845. *Proposed by Richard L. Breisch, Pennsylvania State University.*

Do there exist two 3×3 magic squares A and B, each containing the digits 1 through 9 such that

$$\det A + \det B = \det(A + B)?$$

846. *Proposed by Alfred Kohler, Long Island University, New York.*

Baron Georges Hausmann has decided to reconstruct the streets of Paris so that for every set of three important points in the city which happen to lie on a straight line, a boulevard joining these points is to be constructed. The treasury and the post office are on Rue Madrid, the opera house and police headquarters are on Rue Roma, and the railroad terminal and palace are on Rue Berlin: All three of these streets radiate from Étoile in straight lines. The Baron has already ordered the construction of boulevards joining the treasury, the opera house and the theater; the post office, the police headquarters and the theater; the palace, the police headquarters and the cathedral; the railroad terminal, the treasury and the food market; and the palace, the post office and the food market.

The Baron now intends to issue an order for the construction of a boulevard joining the food market, the cathedral and the theater since he thinks that these three points also lie on a straight line. The owners of the houses lying along the proposed route are opposed to the new boulevard, and they maintain that these points do not lie along a straight line after all. Resolve this dispute by determining whether or not the food market, the cathedral and the theater do lie along a straight line.

847. *Proposed by R. Shantaram, University of Michigan-Flint.*

Prove that:

$$e^{\pi/4} < 1 + 4/\pi.$$

848. *Proposed by J. Prasad, University of Dar Es Salaam, Tanzania.*

Prove that:

$$\sum_{s=0}^{n+1} (r+s)!/s! = \binom{n+r}{n-r} r!,$$

$r = 1, 2, 3 \cdots n$.

849. *Proposed by William Wernick, City College of New York.*

If PB is a line perpendicular to the plane of the triangle ABC, find the necessary and sufficient conditions that angle $APC <$ angle ABC.

850. *Proposed by Richard Dykstra, University of Missouri-Columbia.*

If $a_0 > 0$, $0 < \alpha < 1$, and $a_i = a_{i-1} + a_{i-1}^{\alpha}$, $i = 1, 2, 3 \cdots$ for what values of β will $\sum_{k=1}^{n} (1/a_k)^{\beta}$ converge?

851. *Proposed by D. Rameswar Rao, Sitafalmandi, Secunderabad, India.*

If $p = 2k + 1$ is a prime number and p does not divide $a - 1$, a or $a + 1$, then show that p divides $(a^{2k} - 1)/a^2 - 1$.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q552. Find the greatest possible number of intersections of the diagonals of a plane convex polygon having n sides.

[*Submitted by Rodney T. Hood*]

Q553. Suppose y, s are integers such that $y \neq s$ and $(y, p) = 1$ for prime integer p. Then for integers t and k $(k > 0)$, $yt \equiv 1 \pmod{p^k}$ and $st \not\equiv 1 \pmod{p^k}$.

[*Submitted by George D. Poole*]

Q554. The interior of a circle contains two million points such that no pair of these points lie on any diameter. Must there exist a diameter which contains on each side of it exactly one million of the points?

[*Submitted by Erwin Just and Norman Schaumberger*]

Q555. Show that the sum of n consecutive integers plus n^2 is equal to the sum of the next n consecutive integers.

[*Submitted by Charles W. Trigg*]

Q556. What is the average number of tosses to reach a decision in "Odd Man" (three persons each tossing a coin)?

[*Submitted by John M. Howell*]

(Answers on page 296)

SOLUTIONS

Late Solutions

M. T. Bird, San Jose State College: **814.**

Not So Forbidding Cryptarithm

817. [January, 1972] *Proposed by Charles W. Trigg. San Diego, California.*

In the following cryptarithm, each letter represents a distinct decimal digit, $3(B\ I\ D\ F\ O\ R) = 4(F\ O\ R\ B\ I\ D)$. Find the digits.

Solution by Elizabeth Anne Calog, Northern Michigan University.

Given $3(BIDFOR) = 4(FORBID)$, we can express the "numbers" as $3[1000(BID) + (FOR)] = 4[1000(FOR) + (BID)]$. Removing brackets we get $3000(BID) + 3(FOR) = 4000(FOR) + 4(BID)$. Collecting "like terms" we have $2996(BID) = 3997(FOR)$. Now dividing through by seven we get $428(BID) = 571(FOR)$, or $\frac{428}{571} = \frac{(FOR)}{(BID)}$. Since $\frac{428}{571}$ is the only form of this fraction with three-digit numerals in both numerator and denominator, the solution is unique. Thus, $B = 5$, $I = 7$, $D = 1$, $F = 4$, $O = 2$, $R = 8$, and $3(BIDFOR) = 4(FORBID)$ becomes $3(571{,}428) = 4(428{,}571)$ or $1{,}714{,}284 = 1{,}714{,}284$.

Also solved by Philip H. Anderson, Montclair State College; Carl A. Argila, De La Salle College, Manila, Philippines; V. S. Blanco, University of South Alabama; Dermott A. Breault, Microsystems Technology Corp., Burlington, Massachusetts; Richard L. Breisch, Pennsylvania State University; William Cavanaugh, Elkhart, Indiana; Ralph E. Edwards, Baltimore Life Insurance Company, Maryland; William C. Fitch, Salem, Oregon; Harry M. Gehman, Buffalo, New York; Larry M. Hopkins, Gogebic Community College, Ironwood, Michigan; Sidney Kravitz, Dover, New Jersey; John W. Milson, Butler Community College, Pennsylvania; William Nuesslein, SUNY-Geneseo; Joseph O'Rourke, Saint Joseph's College, Pennsylvania; C. C. Oursler, Southern Illinois University at Edwardsville; Warren Page, New York City Community College; Albert J. Patsche, Rock Island Arsenal, Illinois; Fred Pence, Harrisonburg, Virginia; Sidney Penner, Bronx Community College; Len Pikaart and J. W. Wilson (jointly), Athens, Georgia; Robert H. Poste, Overland Park, Kansas; Lawrence A. Ringenberg, Eastern Illinois University; Marilyn Rodeen, Balboa High School, San Francisco, California; Rina Rubenfeld, New York City Community College; Donald Sandberg, Peshtigo, Wisconsin; Erwin Schmid, Woodacres, Maryland; Edith V. Sloan, University of North Carolina; Daniel M. Smith, Allegan, Michigan; J. P. Smith, Woodson High School, Fairfield, Virginia; E. P. Starke, Plainfield

New Jersey; Paul Sugarman, Massachusetts Institute of Technology; Jim Tattersall, Attleboro, Massachusetts; Philip Tracy, Liverpool, New York; Wolf R. Umbach, Rottdorf, Germany; Willard Uncapher, Buffalo, New York; R. F. Wardrop, Central Michigan University; M. Carl Weiss, California State Polytechnic University at San Luis Obispo; Lamarr Widmer, University of Iowa; Kenneth M. Wilke, Topeka, Kansas; Sister Mary Frances Willcoxon, Little Chute, Wisconsin; and the proposer.

The Rod in a Hemispherical Bowl

818. [January, 1972] *Proposed by W. K. Viertel, SUNY Agricultural and Technical College, Canton, New York.*

Find the angle of repose of a rod of length d and of negligible thickness, when placed in a hemispherical bowl of diameter d. Assume no friction.

Solution by John H. Staib, Drexel University, Philadelphia.

Very likely this problem is in the polar-coordinate chapter of some calculus text; it should be in every such text for it so neatly illustrates the advantage of introducing a coordinate system, and, of choosing the right coordinate system. To see that this is so consider the following two solutions.

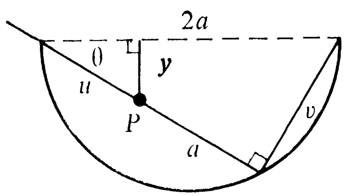


FIG. 1

Solution 1. (Using no coordinate system). This solution depends on our being inventive enough to add to the bowl-rod configuration the additional lines shown in Figure 1. The following equations are then immediate:

$$y = u\sin\theta$$

$$v = 2a\sin\theta$$

$$v^2 + (u + a)^2 = 4a^2.$$

Eliminating u and v leads to the equation

$$y = a(\sin 2\theta - \sin\theta).$$

Since the rod will come to a rest when P — its center of gravity — is at the lowest possible point, we can find the "angle of repose" by finding that value of θ which maximizes y. Setting $y' = 0$ leads to

$$4\cos^2\theta - \cos\theta - 2 = 0,$$

for which we obtain $\cos\theta = (1 \pm \sqrt{33})/8$. But we require an acute angle. Thus, we take the positive value and determine finally that $\theta = 32° \, 32'$.

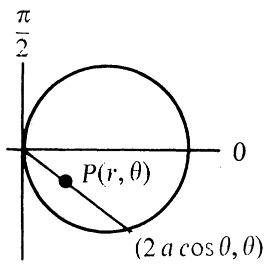


FIG. 2

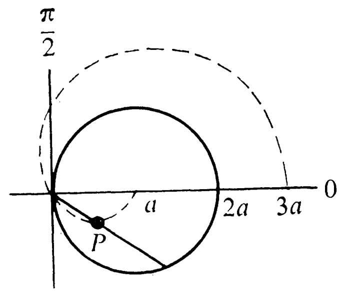


FIG. 3

Solution 2. (Using the polar plane). This solution depends on our being inventive enough to identify the bowl and the rod with the polar configuration shown in Figure 2. It follows immediately that the path of P is described by

$$r = 2a\cos\theta - a, \qquad -\frac{\pi}{2} \leqq \theta \leqq 0,$$

which identifies this motion as being on a limaçon with a loop (see Figure 3). To find the lowest point occupied by P we have occasion to use the formula for the slope of a polar curve:

$$m = \frac{r + r'\tan\theta}{r' - r\tan\theta}.$$

We set $r + r'\tan\theta = 0$, introduce $r(\theta)$ and $r'(\theta)$, and come finally to the same equation as before:

$$4\cos^2\theta - \cos\theta - 2 = 0.$$

The solution of this equation that we seek is $\theta = -32° 31'$, the negative sign being a consequence of our particular use of the polar plane.

This solution has the advantage of not requiring construction lines, of being computationally more elegant, of displaying the actual path of P, and — an extra — of providing an interpretation for the other value of $\cos\theta$.

Also solved by Howard Eves, University of Maine; Robert G. Griswold, University of Hawaii, Hilo College; Michael Goldberg, Washington, D. C.; Larry M. Hopkins, Gogebic Community College, Ironwood, Michigan; Vaclav Konecny and N. J. Vara (Jointly), Jarvis Christian College, Texas; Lew Kowarski, Morgan State College, Maryland; Jan B. Schipmolder, University of California, San Diego; William Nuesslein, SUNY-Geneseo; R. C. Simpson, St. Lawrence University, New York; V. K. Viertel, SUNY-Canton.

Expressing a Finite Sum

819. [January, 1972] *Proposed by Elkedagmar Heinrich, Frankfurt, Germany.*

Find a formula for $\sum_{n=0}^{m} n^k$, k any positive integer.

I. *Solution by Edward T. Frankel, Schenectady, New York.*

$$\sum_{n=0}^{m} n^k = d_1 \binom{m+k}{k+1} + d_2 \binom{m+k-1}{k+1} + \cdots + d_k \binom{m+1}{k+1},$$

where $d_r = \binom{k+1}{0} r^k - \binom{k+1}{1}(r-1)^k + \cdots \pm \binom{k+1}{r-1} 1^k$, the terms being alternately positive or negative.

This formula is a special case of Frankel's general formula [1] for iterated summation of any polynomial sequence of the kth degree.

The d's are the "parallel leading differences" of the sequence $0^k, 1^k, 2^k, \cdots m^k$. They are sometimes referred to as Eulerian Numbers [2].

References

1. E. T. Frankel, A calculus of figurate numbers and finite differences, Amer. Math. Monthly, 57 (1950) 20, 21, 24, 25.
2. John Riordan, An Introduction to Combinatorial Analysis, Wiley, New York, 1958, pp. 39, 215.

II. *Solution by Michael Goldberg, Washington, D.C.*

The sum $\sum_{n=0}^{m} n^k \equiv \sum_{n=1}^{m} n^k$ = the sum of the kth powers of the first m digits = an integral function of the $(k+1)$th degree in m. A method of obtaining this function, for any value of k, is given on pages 486–487 of Vol. 1 of *Text Book of Algebra*, G. Chrystal.

A general expression for this function was obtained by James Bernoulli in terms of the Bernoulli numbers B_i, and it is given on Page 233 of Vol. 2 as

$$\sum_{n=1}^{m} n^k = \frac{m^{k+1}}{k+1} + \frac{m^k}{2} + \frac{k}{2!} B_1 m^{k-1} - \frac{k(k-1)(k-2)}{4!} B_2 m^{k-3}$$
$$+ \frac{k(k-1)(k-2)(k-3)(k-4)}{6!} B_3 m^{k-5}$$
$$- \cdots$$

the last term being $(-1)^{(k-2)/2} B_{k/2} m$ or $\frac{1}{2}(-1)^{(k-3)/2} k B_{(k-1)/2} m^2$ according as k is even or odd.

Also solved by Wray Brady, Slippery Rock State College, Pennsylvania; Richard L. Breisch, Pennsylvania State University; Howard Eves, University of Maine; Edward G. Frankel, Schenectady, New York; David F. Hayes, San Bruno, California; James C. Hickman, University of Iowa; John M. Howell, Littlerock, California; Vaclav Konecny, Jarvis Christian College, Texas; P. A. Lindstrom, Genesee Community College, New York; Paul A. Marchitelli, Gray, New York; J. P. Smith, Woodson High School, Fairfax, Virginia; R. S. Stacey, Albuquerque, New Mexico; Kenneth M. Wilke, Topeka, Kansas; J. W. Wilson, Athens, Georgia; Charles Ziegenfus, Madison College, Virginia; and the proposer.

A number of references were given to the problem. Frankel pointed out his article *A calculus of figurate numbers and finite differences*, AMERICAN MATHEMATICAL MONTHLY, January, 1950; Wilke found the problem in Dorries, 100 *Great Problems of Elementary Mathematics;* Wilson found the problem in Polya, *Mathematical Discovery* and Ziegenfus found it in LeVeque, *Topics in Number Theory.*

Five Full Houses!

820. [January, 1972] *Proposed by Ronald Alter, University of Kentucky.*

Given an ordinary poker deck of 52 cards, choose 25 cards from the deck at random.

a. What is the probability of getting five full houses?

b. What is the maximum number of different arrangements of five full houses that could occur?

Solution by Lawrence A. Ringenberg, Eastern Illinois University.

a. The number of different subsets of 5 ranks, from 13 ranks, is $C(13,5) = 1287$. The number of different subsets of 15 cards, from 52 cards, consisting of 5 triples (threes of a kind) is $4^5 \cdot 1287 = 1317888$.

Given 8 ranks (32 cards), the number of different subsets of 10 cards consisting of 5 pairs of different rank is $6^5 \cdot C(8,5) = 435456$, the number of different subsets of 10 cards consisting of a quadruple (four of a kind) and 3 pairs of different rank is $6^3 \cdot C(8,1) \cdot C(7,3) = 60480$, the number of different subsets of 10 cards consisting of two quadruples and a pair is $6 \cdot C(8,2) \cdot C(6,1) = 1008$, and the number of different subsets of 10 cards that form 5 pairs is 496944.

The number of different subsets of 25 cards, from 52 cards, that can be arranged into 5 full houses is $1317888 \cdot 496944 = 654916534272$.

The probability of getting 25 cards that form 5 full houses is

$$\frac{654916534272}{C(52,25)} \approx 0.0013714.$$

In other words, the chances are about one in a thousand.

b. The number of different arrangements of 5 triples is $1317888 \cdot 120$. Given 32 cards in 8 ranks, the number of different arrangements of 5 pairs of different ranks from those 32 cards is $435456 \cdot 120$, the number of different arrangements of 5 pairs with 2 pairs of the same rank and 3 pairs of different rank is $60480 \cdot 6 \cdot 120$, the number of different arrangements of 5 pairs with two pairs of one rank and two pairs of another rank is $1008 \cdot 36 \cdot 120$, and the number of different arrangements of 5 pairs is $834624 \cdot 120$. The maximum number of different arrangements of five full houses is $1317888 \cdot 120 \cdot 834624 \cdot 120 \approx 1.5839 \times 10^{16}$.

Comment. If five hands of five cards are dealt from a poker deck, the probability that each hand is a full house is about $\dfrac{1.5839 \cdot 10^{16} \cdot 120^5 27!}{52!} \approx 5.3212 \times 10^{-14}$, that is, the chances are about one in twenty trillion, or assuming a million draw poker five-hand deals weekly, about once every 400,000 years on the average.

Also solved by Kenneth Bullis and George Benditt (Jointly), Westmont College, Santa Barbara, California; William Nuesslein, SUNY-Geneseo; and Robert S. Stacey, Albuquerque, New Mexico.

A Compass Construction

821. [January, 1972] *Proposed by Rina Rubenfeld, New York City Community College.*

A and B are two given points. Looking at them as two vertices of a square, find the other two vertices of the square using a compass only.

I. *Solution by Howard Eves, University of Maine.*

Three well-known "compass" constructions are: I. Given points A and B, to find C such that B is the midpoint of AC; II. Given points A and B, to find the midpoint M of AB; III. Given a semicircular arc of diameter AB and center M, to find the midpoint U of the arc. Following are solutions of these three problems, in the standard notation for such constructions (see, e.g., Howard Eves, *A Survey of Geometry*, revised edition, p. 169 ff).

$$\text{I.}\quad \begin{array}{c|c|c} B(A),\ A(B) & B(A),\ D(B) & B(A),\ E(B) \\ \hline D & E & C \end{array}.$$

II. Having found C as in I, proceed as follows:

$$\begin{array}{c|c} C(A),\ A(B) & X(A),\ Y(A) \\ \hline X,\ Y & M \end{array}.$$

$$\text{III.}\quad \begin{array}{c|c|c|c} M(A),\ A(M) & M(A),\ R(M) & A(S),\ B(R) & M(A),\ A(MT) \\ \hline R & S & T & U \end{array}.$$

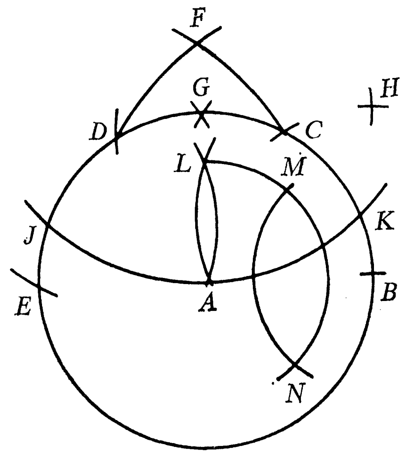
We now turn to the proposed problem. If A and B are consecutive vertices of the sought square, first find C such that B is the midpoint of AC (by I), then the midpoint U of the semicircular arc of diameter AC and center B (as in III). Now find V as an intersection of $U(B)$ and $A(B)$. Then $ABUV$ is the sought square.

If A and B are diagonally opposite vertices of the sought square, first find the midpoint M of AB (by II), and next the midpoints U and V of the two semicircular arcs on AB as diameter (by III). Then $AUBV$ is the sought square.

II. *Solution by Michael Goldberg, Washington, D.C.*

Draw the circle, designated by $A(AB)$, whose center is A and whose radius is AB. Step off the radius along this circle to obtain the points C, D, E, where $BC = CD = DE = AB$. Then, $BD = \sqrt{3}(AB)$. Draw arcs $B(BD)$ and $E(BD)$ intersecting in F. Then, $AF = \sqrt{2}(AB)$. Draw arcs $B(AF)$ and $E(AF)$ intersecting in G which lies on the circle $A(AB)$. If A and B are adjacent vertices of the sought square, then G is a third vertex. The fourth vertex H is the intersection of $G(AB)$ and $B(AB)$.

If A and B are opposite vertices of the sought square, then find L which is the inverse of F with respect to the circle $A(AB)$. This is performed by drawing the arc $F(AF)$ which intersects $A(AB)$ in J and K, and then drawing arcs $J(AJ)$ and $K(AK)$ which intersect in L. Then, $AL = (AB)/\sqrt{2}$. Hence, the other vertices M and N of the sought square are the intersections of $A(AL)$ and $B(AL)$.



Also solved by Wray G. Brady, Slippery Rock State College, Pennsylvania; Thomas M. Fielstra, Muskegon, Michigan; William C. Fitch, Salem, Oregon; David F. Hayes, San Bruno, California; Brian Hill, Grinnell College, Iowa; Vaclav Konecny, Jarvis Christian College, Texas; Peter A. Lindstrom Genesee Community College, New York; Raymond A. Maruca, Delaware County Community College, Media, Pennsylvania; William Nuesslein, SUNY-Geneseo; Ronald E. Rossi, De Anza College, Cupertino, California; Warren Page, New York City Community College; E. P. Starke, Plainfield, New Jersey; J. W. Wilson, Athens, Georgia; William Wernick, City College of New York; Dick A. Wood, Seattle Pacific College, Washington; and the proposer.

An Inequality

822. [January, 1972] *Proposed by C. S. Venkataraman, Sree Kerala Varma College, Trichur, South India.*

Given a_k $(k = 1,2,\cdots,n)$ is zero or any positive integer and A_n is their arithmetic mean, prove that $\prod_{k=1}^{n} a_k! \geqq [A_n]^n!$ with equality arising when $n = 1$ or when all the a's are equal, and where $[x]$ is the integral part of x.

Solution by N. J. Kuenzi and Bob Prielipp (jointly), Wisconsin State University at Oshkosh.

We believe there was a misprint in the statement of the problem. It is not difficult to exhibit an example for which

$$\prod_{k=1}^{n} a_k! < [A_n]^n!.$$

We believe that the inequality should have been

$$\prod_{k=1}^{n} a_k! \geqq ([A_n]!)^n$$

and so we made this change in the statement of the problem. There is one other small point — equality can also arise when one has a sequence of 0's and 1's as is seen in our proof.

We will induct on the difference between the largest value and the smallest value of the a_k's. Without loss of generality we will assume that $0 \leqq a_1 \leqq \cdots \leqq a_n$.

If $a_n - a_1 = 0$, then $a_1 = A_n = a_n$ and $\prod_{k=1}^{n} a_k! = (A_n!)^n$.

If $a_n - a_1 = 1$, then $a_1 < A_n < a_n = a_1 + 1$. Therefore, $[A_n] = a_1$ and $\prod_{k=1}^{n} a_k! \geqq ([A_n]!)^n$ with equality holding if and only if $a_1 = 0$ and $a_n = 1$.

Assume that $\prod_{k=1}^{n} a_k! \geqq ([A_n]!)^n$ whenever $a_n - a_1 \leqq m$ and consider the case when $a_n - a_1 = m + 1$.

Form a new sequence $a_1^*, a_2^*, \cdots a_n^*$ as follows: $a_j^* = a_j + 1$ and $a_{n-j+1}^* = a_{n-j+1} - 1$ if $a_{n-j+1} - a_j = m + 1$, $a_j^* = a_j$ otherwise.

Note that $a_1!a_n! > a_1^*!a_n^*!$, $a_j!a_{n-j+1}! \geqq a_j^*!a_{n-j+1}^*!$ and $a_j + a_{n-j+1} = a_j^* + a_{n-j+1}$. Therefore

$$\prod_{k=1}^{n} a_k! > \prod_{k=1}^{n} a_k^*! \text{ and } A_n = A_n^*.$$

Also note that the difference between the largest value and the smallest value of the a_k^*'s is either $m - 1$ or m.

Therefore,

$$\prod_{k=1}^{n} a_k! > \prod_{k=1}^{n} a_k^*! \geqq ([A_n]!)^n.$$

Also solved by Vaclav Konecny, Jarvis Christian College, Texas; William Nuesslein, SUNY-Geneseo; Philip Tracy, Liverpool, New York; Kenneth M. Wilke, Topeka, Kansas; E. P. Starke, Plainfield, New Jersey; and the proposer.

Euler's ϕ-Function

823. [January, 1972] *Proposed by Ira Gessel, Dayton, Ohio.*

Find a function f such that $\sum_{k=1}^{m} f(k)[m/k] = m(m+1)/2$.

Solution by Vaclav Konecny, Jarvis Christian College, Texas.

Subtracting $\sum_{k=1}^{m} f(k)[m/k] = m(m+1)/2$ and

$$\sum_{k=1}^{m-1} f(k)[(m-1)/k] = (m-1)m/2$$

we get

$$f(m) + \sum_{k=1}^{m-1} f(k)\{[m/k] - [(m-1)/k]\} = m.$$

It is easy to prove that

$$[m/k]-[(m-1)/k] = \begin{cases} 1 & \text{if } m \text{ is divisible by } k \\ 0 & \text{if } m \text{ is not divisible by } k. \end{cases}$$

Thus $f(m)+\sum_k f(k) = m$ where we add over all divisors of $m\,(k<m)$ or $\sum_k f(k) = m$, $(k \leqq m)$.

Therefore $f(k) = \phi(k)$ where ϕ is Euler's ϕ-function.

The problem may be found (there is a hint for solving it in another way) in *Elementary Number Theory*, by J. V. Uspensky and M. A. Heaslet, McGraw-Hill, New York, 1939, p. 116.

Also solved by M. T. Bird, San Jose, California; Ira Gessel, Dayton, Ohio; Fred Dodd and Leon E. Mattics (Jointly), University of South Alabama; John Gilroy, Iowa City, Iowa; Robert Guy, Framingham State College, Massachusetts; James C. Hickman, University of Iowa; Arthur Marshall, Madison, Wisconsin; Kenneth Ribet, Cambridge Massachusetts; Jan B. Schipmolder, University of California at San Diego; Joseph Simone, University of Missouri at Kansas City; R. S. Stacy, Albuquerque, New Mexico; E. P. Starke, Plainfield, New Jersey; Jim Tattersall, Attleboro, Massachusetts; Philip Tracy, Liverpool, New York; Kenneth M. Wilke, Topeka, Kansas; Paul J. Zwier, Calvin College, Michigan; and the proposer.

Comment on Problem 768

768. [September, 1970, and May, 1971] *Proposed by J. A. H. Hunter, Toronto, Canada.*

Solve the alphametic

$$\begin{array}{r} A \\ G\ O \\ G\ O \\ G\ A\ L \\ \hline L\ O\ O\ K \end{array}$$

Comment by Edwin P. McCravy, Midlands Technical Education Center, Columbia, South Carolina.

A simple matter was overlooked. The solution published was $A = 4$, $G = n-2$, $K = 5$, $L = 1$, $O = 0$, "for a base $n \geqq 6$". But if $n = 6$, then $A = G = 4$; or if $n = 7$, then $G = K = 5$. These equalities are not permitted in view of the stated condition that "different letters represent distinct digits". Thus the condition should have been "$n \geqq 8$" not 6.

Comment on Problem 775

775. [November, 1970, and September, 1971] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Prove $\int_0^1 \sqrt[q]{1-x^p}\,dx = \int_0^1 \sqrt[p]{1-x^q}\,dx$, where $p,q > 0$.

Comment by Ralph Leung, Berkeley, California.

The problem would become immediate if we rewrite the above equality as

$$\int_0^1 \sqrt[q]{1 - x^p}\,dx = \int_0^1 \sqrt[p]{1 - y^q}\,dy.$$

Both sides give the area of the region bounded by the x-axis, the y-axis, and the graph of $x^p + y^q = 1$ in the first quadrant — the l.h.s. by integrating with respect to x, the r.h.s. with respect to y.

Comment on Problem 793

793. [March, 1971, and January, 1972] *Proposed by Gregory Wulczyn, Bucknell University.*

Find the determinant of lowest order with entries from the interval $-1 \leqq x \leqq +1$ whose value is 2^{38}.

Comment by Joel Brenner and Larry Cummings (jointly), University of Waterloo, Canada.

Our article on the maximal determinant problem has appeared in the AMERICAN MATHEMATICAL MONTHLY research problem section [**1**]. Since there is a slight misprint on page 54 in the solution to problem 793 you may wish to publish a correction, or a reference to our article.

Using the notation of your journal, the article of Ehlich shows that $g(18) = 17 \cdot 2^{33}$; the solution should read $g(18) = 17 \cdot 2^{33} < 2^{38} < 20^9 \leqq g(19)$ (not $\geqq g(19)$).

If your proposer had replaced "interval $-1 \leqq x \leqq 1$" in his problem by "the set $-1, 1$" the answer would be unknown up to the present day. It is known, however, that the determinant of every $n \times n$ matrix of -1's and 1's is divisible by 2^{n-1}. This is proved by simple manipulations on the rows and columns. Thus the determinant of any 19×19 matrix of -1's and 1's is divisible by 2^{18}.

References

1. J. Brenner and L. Cummings, The Hadamard maximum determinant problem, Amer. Math. Monthly, 79 (1972) 626–630.

2. Harmut Ehlich, Determinantenabschätzungen für binäre Matrizen, Math. Z., 83 (1964) 127, Satz 4.2.

Comment on Q516

Q516. [March, 1971] Show that if a, b, and c are any positive real numbers, then

$$c \geqq 3\sqrt[3]{abc} - 2\sqrt{ab}.$$

[*Submitted by Norman Schaumberger*]

Comment by W. S. Hall, University of Pittsburgh.

When $x,y \geqq 0$, $p+q=1$, then $x^p y^q \leqq px+qy$. (Special case of Young's inequality.) Hence,

$$3\sqrt{abc} = (ab)^{(1/2)(2/3)}c^{(1/3)} \leqq (2/3)\sqrt{ab} + (1/3)c$$

and the result follows. The inequality used above can be easily derived by noting that for positive x, $d(\ln x)/dx > 0$ and $d^2(\ln x)/dx^2 < 0$. Hence $\ln x$ is monotone increasing and concave. Therefore $\ln(px+qy) \geqq p\ln x + q\ln y = \ln x^p y^q$. Thus, $x^p y^q \leqq px+qy$.

Comment on Q530

Q530. [November, 1971] Suppose that $a-1$ and $a+1$ are twin primes larger than 10. Prove that a^3-4a is divisible by 120.

[*Submitted by George E. Andrews*]

Comment by Charles W. Trigg, San Diego, California.

For a stronger result, note that $(a-2)(a-1)a(a+1)(a+2)$ is the product of five consecutive integers, so one of them is divisible by 3 and one is divisible by 5. If $(a-1)$ and $(a+1)$ are primes, then $(a-2)$, a, and $(a+2)$ are consecutive even integers, so at least one of them is divisible by 4, and each of the others is divisible by 2. Hence the product of the three integers adjacent to twin primes > 5 is divisible by 3(5)(4)(2)(2). That is, $240 | (a^3-4a)$. Indeed, if a is twice an odd integer, say 42, then $480 | (a^3-4a)$. In the latter case, the twin primes are $6k-1$ and $6k+1$ with k an odd integer.

Comment on Q532

Q532. [January, 1972] Without evaluating either of the following integrals show why

$$2\int_{-1}^{1}\sqrt{1-x^2}\,dx = \int_{-1}^{1} 1/\sqrt{1-x^2}\,dx.$$

[*Submitted by Peter A. Lindstrom*]

Comment by Vaclav Konecny, Jarvis Christian College, Texas.

It is also evident that it is the same as to show

$$2\int_0^1 u^{-1/2}(1-u)^{1/2}du = \int_0^1 u^{-1/2}(1-u)^{-1/2}du \text{ or}$$

$2B(1/2, 3/2) = B(1/2, 1/2)$ which is indeed true. (B is the beta function.)

ANSWERS

A552. Any four vertices determine a complete quadrilateral. Exactly one diagonal point of this quadrilateral lies within the polygon (because of its convexity). This point lies on just two diagonals. Therefore, the greatest number of intersections of diagonals of the n-gon occurs when these are all distinct. This number is equal to the number of such quadrilaterals, or

$$nC_4 = \frac{n(n-1)(n-2)(n-3)}{24}.$$

A553. Choose k such that $0 < |y - s| < p^k$. Since $(y, p) = 1$, the set P of elements t such that $yt \equiv 1 \pmod{p^k}$ is not empty. Any t in P satisfies $st \equiv 1 \pmod{p^k}$.

A554. Choose a diameter which contains none of the points of the given set and assume that the interior of one of the semicircles determined by some orientation of this diameter contains precisely $1{,}000{,}000 - k$ points. If the diameter is rotated 180°, then the interior of the semicircle determined by the same orientation of the diameter will contain precisely $1{,}000{,}000 + k$ points. Since the hypothesis requires that a rotating diameter pass through precisely one point at a time, it follows that the diameter must have assumed at least one position in which the interior of each of its semicircles contained precisely one million points.

A555. Using the formula for the sum of an arithmetic progression:

$$\begin{aligned}\sum_{k+1}^{k+n} i + n^2 &= n(2k+n+1)/2 + 2n^2/2 \\ &= n(2k+3n+1)/2 \\ &= \sum_{k+n+1}^{k+2n} i.\end{aligned}$$

A556. Using

$$\begin{aligned}\sum_{x=1}^{\infty} xar^{x-1} &= 1/(1-r) \sum_{x=1}^{\infty} ar^{x-1} \\ &= a/(1-r)^2\end{aligned}$$

we have

x	1	2	3	$\cdots$	n
$p(x)$	3/4	3/4(1/4)	$3/4(1/4)^2$	$\cdots$	$3/4(1/4)^{n-1}$

or $3/4\, xp(x) = 4/3$.

(Quickies on pages 284–285)

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The Fifth Annual Greater Metropolitan New York Mathematics Fair will be held at Pace College, New York City, on March 11, 1973. The purpose of the Fair is to enable the secondary school student to pursue, in depth, some phase of mathematice of interest to him, and to present a paper on this topic to a panel of judges. Although the paper need not be original mathematics, it should reflect scholarship appropriate to the course level of the student. Judges both from collegiate and secondary school faculties are also needed.

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ACKNOWLEDGMENT

The editors wish to thank the following mathematicians for their assistance in refereeing manuscripts submitted to the MATHEMATICS MAGAZINE: G. Angwin, R. A. Askey, G. E. Baxter, J. A. Beidler, B. Berndt, E. E. Burniston, I. Burr, L. Cote, F. S. Cotter, R. Crittenden, E. Cumberbatch, D. Drasin, M. Drazin, U. Dudley, E. Ecklund, J. M. Gandhi, C. Goffman, M. Golomb, D. Gottlieb, H. Gould, R. Hansen, J. H. Jordan, M. L. Keedy, P. J. Kelly, J. Levine, C. T. Long, R. E. Lynch, E. A. Maier, R. H. Martin, Jr., T. W. Mullikin, C. Neugebauer, M. Neuts, H. Niederreiter, L. Nyhoff, S. Perlis, H. Rubin, J. E. Rubin, S. Samuels, E. Schenkman, J. Selfridge, M. E. Shanks, R. Silber, E. Silverman, R. Spira, R. Stalley, R. A. Struble, D. Thoe, J. K. Thurber, D. F. Ullrich, C. L. Vanden Eynden, W. A. Webb, S. Weingram, G. Wheatley, J. Yackel, P. Young.

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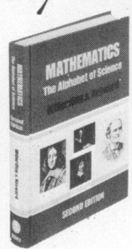

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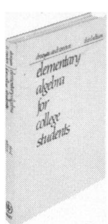

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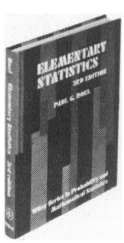

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